Managing Hedge Fund Liquidity Risks

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Abstract

We study hedge fund liquidity management in the presence of liquidity risks on the asset and liability sides. We formulate a two-period model where a single fund has always access to a liquid asset and can invest in an illiquid asset which pays off only at the end of period two. Funding liquidity risk takes the form of a random outflow originating from clients in period one. The fund suffers from a random haircut on the illiquid asset's secondary market to cover its outflow. We solve the allocation problem of the fund and find its optimal allocation between liquid and illiquid assets. We show that the liquidation probability and the portfolio composition of the fund are revealing about the market liquidity and funding liquidity, respectively. Gates, as a device that limits the outflows experienced by the fund, helps it reduce its liquidation risk and harvest liquidity premia.

JEL Codes: G11, G12.

Key-words: Risk management, hedge fund, market liquidity, funding liquidity, default probability.

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1 Introduction

Hedge fund managers select investment opportunities for their clients to generate the best possible performance. One such way to chase performance is to target assets that suffer from illiquidity, that is assets that are relatively cheaper because they are harder to resell on a secondary market before their cashflows are realized. While it may seem optimal for fund managers to fully invest into these assets to capture the illiquidity premium and offer high returns to their clients, they rarely implement it in practice: in the few cases where funds offer transparency on their portfolios, a significant share of assets under management is invested in cash or in liquid assets with lower expected returns. One of two things can prevent the manager to capture the liquidity premium: either agency problems are diverting her to maximize their clients' returns, or her portfolio achieves optimal risk management given the remaining of the fund's balance sheet, beyond the asset side. Since most funds are open-ended, thereby allowing clients to withdraw their capital on a short horizon[2], the latter explanation seems to be of first order, such that the fund is actively managing liquidity risks on both the asset and liability sides. Indeed, failure to reimburse clients triggers the liquidation of the fund, an event that has been understudied in the hedge fund asset pricing literature. Conversely, empirical papers have shown that funds invested in illiquid assets capture a significant premium, leaving aside the idea that the liquidity exposure is endogenous to the potential liquidity mismatch between the fund's asset and liability sides, potentially underestimating the liquidity premium. The funds that are naturally more exposed to large funding and/or market liquidity shocks through high discretionary liquidity restrictions (DLR) for instance will endogenously adjust their portfolios to avoid liquidation.

This paper provides a framework to understand the trade-offs of a fund manager based on the liquidity characteristics of his portfolio (assets) and his investors (liability).[3] We formulate a two-period model where the fund manager is endowed with one unit of cash that she can invest at time zero in either a zero-interest liquid asset (cash henceforth), or in an illiquid asset that generates a positive deterministic return at period two. In our model, liquidity risks are subordinated. The primary risk faced by the fund is that clients withdraw their assets under management at period 1, forcing the fund to raise cash on the asset side to finance the outflow. If the fund's already available cash is not sufficient to cover the outflow, it can start selling its illiquid asset on a secondary market, where the illiquidity is represented by a random haircut. The manager is forced to liquidate the

[2]Liquid hedge funds are particularly exposed to this risk as these offer daily liquidity while they continue to significantly invest in illiquid assets to harvest the liquidity premium.

[3]Liquidity transformation is often a topic related to banks. In our case, funds act as a bank, but the main difference is in the absence of permanent capital to cover liquidity transformation, and in the absence of specific regulation to frame this risk.

fund whenever the effective selling price of the illiquid asset is such that it cannot cover the capital outflow even by liquidating all its portfolio. Our model can thus generate endogenous liquidation events that arise from the impossibility for the fund manager to reimburse its clients. It corresponds to the conjunction of a high outflow on its liability side and/or poor market conditions to liquidate the risky assets on its balance sheet.

In its baseline version, the model gathers three parameters: the probability of having an outflow of assets under management, the average rebate on the secondary market, and the dependence between the return of the illiquid asset in period two and the market liquidity conditions, or liquidity premium parameter. The hedge fund chooses its optimal amount of cash to maximize its terminal value taking these parameters as given. A particular advantage of our framework is that, for any set of parameters and a given cash amount, both the expected value of the fund's portfolio and the fund's liquidation probability are computable in closed form. This allows us to shed light on the link between hedge funds expected returns and liquidation probabilities. After deriving the results in the baseline case, we consider two extensions where *(i)* funds can impose gates that cap the maximum withdrawal they have to face, and *(ii)* the amount of risky asset that fund must liquidate feeds back onto a supplementary haircut, representing a potential market impact.

Our model is directly inspired by the *limits of arbitrage* literature, which justifies that arbitrageurs do not automatically take advantage of evident arbitrage opportunities when their funding is limited. We build around this idea to relate expected returns and liquidation probability of the fund via the optimal liquidity policy. The optimal liquidity level is fundamentally determined by this trade-off between expected returns and liquidation. It is therefore possible to use these two *observed* dimensions to infer the effective *unobserved* liquidity level of a given fund.

Solving for the equilibrium produces three main insights. First, we show that the fund manager naturally shifts her portfolio towards cash whenever liquidity conditions become poorer, irrespective of the side of the balance sheet that is affected. The equilibrium portfolio of the fund consists in making its liquidation probability insensitive to its average expected outflows, while the optimal invested amount of liquid asset is insensitive to the market liquidity conditions. This emphasizes that liquidation probabilities of hedge funds can reveal conditions about liquidity exposure on the asset side, whereas portfolio composition can reveal liquidity exposure on the liability side, contrary to the intuition. These comparative statics have implications on the fund's expected returns. When funding liquidity becomes poorer, the fund adjusts its cash holding upwards such that its liquidation probability stays the same and generates lower expected returns because of the portfolio composition shift. When market liquidity becomes poorer, the fund has a larger probability of liquidation that is compensated by an additional expected return from the illiquid asset. The latter effect dominates the former, such that expected returns increase with market illiquidity, as the empirical evidence suggests. In the end, simultaneous worsening in funding and market liquidity conditions can potentially dampen the ability to identify meaningful movements in hedge funds expected returns due to active liquidity management.

Second, we show that introducing gates that would limit the ability of investors to withdraw their assets under management can largely mitigate these effects, by decreasing default probabilities and then allowing the fund to capture the liquidity premium more efficiently. Whatever funding and market liquidity conditions, larger gates allow the hedge fund to reduce its liquidation probability and increase its expected return. The mechanism differs depending on liquidity conditions. When the probability to experience outflows is large, introducing gates directly decreases the fund's liquidation probability for a given level of cash, more so if market liquidity is poor. It becomes optimal for the manager to decrease its cash holdings to capture the liquidity premium of the risky asset and increase its expected returns. When the outflow probability is small, the fund has incentives to increase its cash holdings to prevent itself further from liquidation and have a greater chance to harvest the liquidity premium. Despite these alternative risk-return trade-offs, gates unambiguously ease the liquidity management of the hedge fund.

Third, we show that when the fund is large enough to have a market impact, i.e. selling the illiquid asset further decrease its selling price, it unambiguously increases its cash holdings such that both liquidation probabilities and expected returns are very little affected. In this extension of our model, we consider that the haircut observed on the secondary market grows with the additional cash that the fund must raise to cover its outflows. As a result, the fund manager increases its cash holding to curb the average haircut suffered in case of a significant outflow. We conclude that the effects of liquidity spillovers from having larger funds can have a substantial effect on the portfolio composition of funds while affecting the performance very little.

To illustrate the usefulness of our model, we propose an empirical application based on the Lipper-TASS database. We observe a cross-section of funds at the monthly frequency or several management styles that we aggregate in single time series. More specifically, we consider the aggregate outflow probability and the liquidation probability by averaging across funds and use them to compute the optimal cash amount, the implied average market liquidity, and the implied risky asset return. We show that the cash amount and the market liquidity are negatively correlated, as the intuition would suggest, and that the funds significantly hedged their illiquid positions during the financial crisis to avoid massive liquidations. We perform counterfactual experiments that suggest that the introduction of gates would greatly reduce the liquidation probability of funds more than linearly. Alternatively, an increase in liquidity premium increases the attractiveness of the risky asset and implies more risk-taking from the fund, increasing their liquidation probability and decreasing their liquid asset holdings. Last and consistently with the model, liquidity spillovers only have a low effect on the model outputs.

The consequence of market and funding liquidity shocks on Hedge Fund performance has been intensively studied in the academic literature. However, most of these studies consider the two risks separately, and only a few papers focus on the consequence of both market and funding liquidity risks on hedge fund performance and survival.

Sadka (2010, 2012) examines funds' exposure to aggregate market-wide liquidity. He investigates whether liquidity risk is priced in hedge fund returns using different liquidity risk factors and finds that funds with high exposure to aggregate liquidity risk outperform those with low exposure by 6% annually during normal months. However, during periods where liquidity is scarce, these funds with high liquidity risk drastically underperform those with low exposure. Teo (2011) studies the performance of the most liquid hedge funds, i.e., those that offer the shortest lockup and redemption notice periods to their investors. He finds that liquid funds that experience net capital outflows may be forced into fire sales, and suffer lower risk-adjusted returns (by 4.79 percent) than their counterparts with high net inflows. Finally, Jame (2015) uses transaction-level data to study whether hedge funds profit from providing liquidity to the market. He focuses in particular on liquidity provision as a source of hedge fund performance.

Dudley and Nimalendran (2010) show that the funding liquidity risk is priced in hedge fund returns. They first develop a funding liquidity risk factor for hedge funds using the residuals of a regression of futures margins on the implied volatility index (VIX). They then include their factor in classic asset pricing tests such as portfolio sorts and factor model regressions. Liu and Mello (2011) relate funding liquidity risk to funds' cash holdings. The optimal cash holding reflects a trade-off between the reduction of liquidation costs and the increase in returns by holding risky assets. They find that investors fear that other investors may withdraw their capital and force a fire sale. They suggest that redemption risk led hedge funds to hold more cash to resist to funding liquidity shocks. Finally, Aragon and Strahan (2012) use the Lehman Brothers bankruptcy as an exogenous shock to hedge funds' funding liquidity. Lehman Brothers was the prime broker for many hedge funds in 2008. The authors find that the Lehman bankruptcy in September 2008 increased by 50% the default probability of funds using this prime broker during the next year. However, they also stress that it is difficult to empirically measure the impact of funding liquidity risk on hedge funds since their strategies and funding arrangements are jointly chosen. To circumvent this empirical issue, Hombert and Thesmar (2014) develop a model *à la* Shleifer and Vishny (1997). They assume that managers choose a set of contractual features that impact how sensitive capital is to poor performance. Therefore, lockup and notice periods restrict investors' ability to withdraw their capital after a period of poor performance. The authors test the model predictions using self-reported liquidity restrictions, estimated flow-performance sensitivity and a new measure that captures the ability of lockup provisions to retain capital inflows. They find that fund managers effectively choose contractual features to reduce investors' ability to withdraw their capital.

Considering simultaneously funding and market liquidity risks, Agarwal, Aragon, and Shi (2015) focus on funds of hedge funds and propose a measure of the difference (named liquidity gap) between the liquidity of fund of funds' assets (i.e., underlying hedge funds in the fund of funds portfolio) and the liquidity of fund of funds liabilities (i.e., redemption by investors). Funds of funds with larger illiquidity gaps are shown to perform badly during crises and exhibit greater exposure to runs (i.e. some investors redeeming strategically prior to others when funds perform poorly). As in Jame (2015), Franzoni and Plazzi (2013) study whether hedge funds profit from providing liquidity to the market. They find that the level of liquidity provision decreases when funding and market conditions deteriorate. They use the VIX index, the TED spread, and the LIBOR rate as proxies for trading costs since these measures are related to the costs of borrowing or to the tightness of margin requirements. Hedge funds' ability and willingness to provide liquidity during times of lower market liquidity is also related to their own fund flows. Finally, a decline in hedge fund trading predicts a decline in liquidity at the individual stock level.

Our contributions are threefold. First, as in Agarwal, Aragon, and Shi (2015), we consider jointly market and funding liquidity risks, but our model endogenizes the fund manager behavior in terms of liquidity risk hedging. Second, we obtain the fund default probability and the liquidity management cost as a function of the market and funding liquidity conditions. We explain how fund managers adjust cash holdings to immunize their fund against funding liquidity shocks. We confirm the first results obtained by Hombert and Thesmar (2014) on contractual provisions management and explain why default probabilities are not increasing too much during the financial crisis (compared to banks' default probabilities). Third, we build a new liquidity factor (the cash amount) that reflect the implied cost of the liquidity mismatch between assets and liabilities. This factor is depending on both the market and the funding liquidity conditions, and not only market (see e.g. Sadka (2010, 2012)) or funding (see e.g. Fontaine and Garcia (2012)) liquidity conditions.

The paper is organized as follows. Section 2 describes the setup where the hedge fund manages both market and funding liquidity risks. Section 3 describes how the model is solved to get the optimal cash holding level and lists the model implications in a benchmark case. Section 4 generalizes these results to cases where gates and liquidity spillovers are present. Section 5 uses hedge fund data to illustrate the model predictions. Section 6 concludes. All the proofs are gathered in the appendix.

2 The model

We present a model in which fund managers can choose to invest in a liquid asset, or cash in the following to simply the presentation, to deal with funding liquidity shocks. In addition, we assume that funds invest in illiquid assets whose valuation is exposed to market liquidity shocks. Our objective is to derive equilibrium relationships between liquidity conditions on the one hand, and fund's performance and liquidation probability on the other hand. We later use these relationships in the empirical section to estimate unobserved liquidity conditions from historical fund returns and defaults.

2.1 Funding liquidity shocks

We consider a two-period economy with a fund entity that is endowed with one unit of capital at $t = 0$. This unit is contributed by a continuum of investors with measure one.[4]

At period $t = 1$, a *funding liquidity* shock can happen with probability π. If the shock is realized, each investor may redeem its shares despite initially having the intention to stay for two periods with the fund. More specifically, each investor closes its position with probability $\theta \sim \mathcal{U}(0, \overline{\theta})$, where the upper limit $\overline{\theta} \in (0, 1]$ represents any legal or credible limit to the maximum outflows in the fund. The expected outflow suffered by the fund *ex-ante* is thus given by $\frac{\pi\overline{\theta}}{2}$.

Our modeling of funding liquidity is largely inspired from the framework developed by Liu and Mello (2011). However, we add a source of flexibility through the parameter π by considering that funds do not have to suffer positive outflows. By allowing this form of asymmetry in the distribution of outflows, we aim at representing the liquidity of the liability side. When π is close to 1, we come back to the case developed by Liu and Mello (2011), which considers open-ended funds such as money market funds for instance. Instead, when π is close to 0, we can represent closed-ended funds such as hedge funds. The π parameter allows us to measure the impact of the structure of funds' clients on the endogenous liquidity exposure chosen by the fund on the asset side.

[4] Our model in its current form does not consider leverage, i.e. the possibility to invest more than the investors' capital.

2.2 Market liquidity shocks

Faced with a potential outflow of clients, the fund chooses to invest its assets under management at $t = 0$. We consider that it can either keep cash, which pays a zero interest rate and is available every period, or invest in an illiquid asset. The latter provides a gross return R known *ex-ante* only if held until period $t = 2$. As in Shleifer and Vishny (1997), we assume that there is no fundamental risk, such that R is greater than one. However, in case the illiquid asset is sold at $t = 1$ on the secondary market, the fund suffers from a haircut such that the liquidation price is given by a random variable α that belongs to $(0, 1)$ almost surely.

In the following, we assume that:

$$-\log \alpha \sim \mathcal{E}xp(\lambda)\,, \tag{1}$$

such that the expected selling price is $\frac{\lambda}{\lambda+1}$, and the CDF of α is given by $F_\alpha(x) = x^\lambda$. This quantity is decreasing in λ, which represents the expected degree of *market liquidity*. Said differently, the bigger λ, the smaller the expected haircut, and the bigger the liquidation price on the secondary market.

2.3 Fund value

So far, we have remained silent on the timing of events and the reasons for liquidating the risky asset. In our framework, the fund chooses its cash quantity $\delta \in (0, 1)$ at $t = 0$ in order to maximize its portfolio value at $t = 2$. We consider that the value of the fund is given by whatever remains on the asset side.

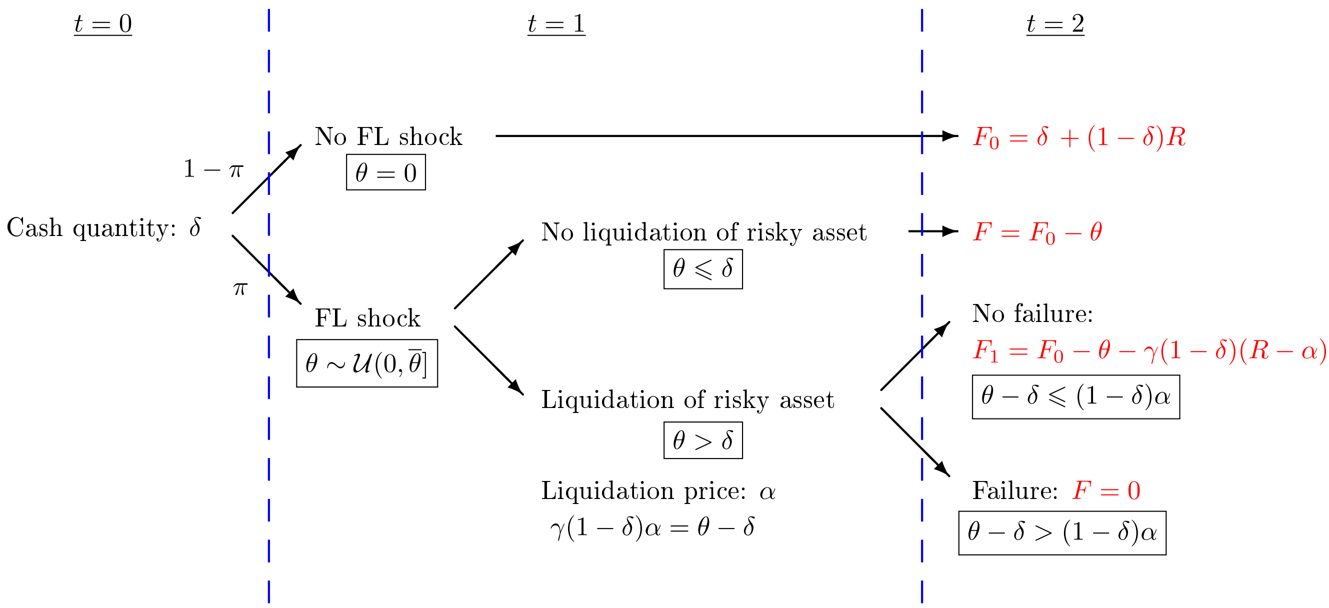
After choosing its portfolio $(\delta, 1 - \delta)$, the fund faces three possible scenarios. First, if there is no outflow on the liability side, the fund's portfolio remains untouched and all potential return of the risky asset is realized. The fund's final value is thus given by $F_0 = \delta + (1 - \delta)R$. A second case arises if clients are withdrawing from the fund, but the latter has a sufficiently large cash cushion to cover all outflows. In this case, the final value of the fund is given by $F_0 - \theta$. The last case arises when the outflows are larger than the cash holdings of the fund, that is when $\theta > \delta$. In this case, the fund starts liquidating its second asset to reimburse the clients.

In this case, the fund sells the minimal amount it needs to cover the outflows. We thus have that the final value of the fund's portfolio in this case is given by (see Appendix A.1):

$$F_1 = \max\left[F_0 - \theta - \gamma(1 - \delta)(R - \alpha), 0\right] \quad \text{where} \quad \gamma(1 - \delta)\alpha = \theta - \delta\,. \tag{2}$$

In Equation (2), we see that whenever positive, the fund's portfolio is equal to its value F_0 absent any shocks, minus two separate costs. The first is θ, the withdrawal from clients that reduces the fund's size. The second is an additional cost linked to the haircut applied on the secondary market when the risky asset is liquidated. We additionally assume limited liability such that the fund's value never go below zero. Last, γ is the proportion of the risky asset holdings needed by the fund to cover the cash shortage $\theta - \delta$. Figure 1 summarizes the setup and the different liquidity scenarios.

Figure 1: Summary of the setup



Notes: FL stands for *funding liquidity.* γ is the quantity of the risky asset that needs to be liquidated to cover the funding liquidity shock θ.

2.4 The fund's maximization problem

We assume that the fund manager's objective is to maximize the fund's total equity value at $t = 2$. As remarked by Liu and Mello (2011), this objective is consistent with the maximization of the asset under management, that is an indicator of the fund's performance and the capacity to attract new investors. Although close in spirit to Hombert and Thesmar (2014), our framework differs in the endogenous choice of the fund. Indeed, they consider that the fund can only endogenously choose the friction to outflows, $\overline{\theta}$ in our notation. In comparison, we assume that funds can keep a liquidity cushion to shelter themselves from funding liquidity shocks. We choose to leave aside the endogenous choice of $\overline{\theta}$, which would require a modeling of the associated costs for the fund. Our approach allows us to represent the day-to-day liquidity management of the fund, whereas it is likely that Gates and barriers to capital outflows are chosen once and for all when the fund is created.

The fund's problem can be formulated as:

$$\begin{aligned} \max_{\delta} \quad & (1-\pi)F_0(\delta) + \pi \cdot \tfrac{\delta}{\overline{\theta}} \left[F_0(\delta) - \tfrac{\delta}{2}\right] \\ & + \quad \pi \cdot \tfrac{\overline{\theta}-\delta}{\overline{\theta}} \cdot \mathbb{E}\left[\max\left(F_0(\delta) - \theta - \gamma(1-\delta)(R-\alpha), 0\right) \,\middle|\, \theta - \delta > 0\right] , \end{aligned} \tag{3}$$

where the three terms correspond to the three possible scenarios where the fund survives. $F_0(\delta)$ emphasizes that the standard value of the fund's portfolio where no liquidity event take place is a decreasing function of δ. After some algebraic manipulation (see Appendix A.2.2), we have:

$$\max_{\delta} \quad F_0(\delta) - \pi \cdot \delta \left(1 - \frac{\delta}{2\overline{\theta}}\right) - \frac{\pi}{\overline{\theta}} \cdot \frac{(\overline{\theta}-\delta)^2}{1-\lambda} \cdot R \cdot \left[\frac{1}{1+\lambda} \cdot \left(\frac{\overline{\theta}-\delta}{1-\delta}\right)^{\lambda-1} - \frac{\lambda}{2}\right] . \tag{4}$$

Three terms naturally emerge from the fund's expected value: a basis value $F_0(\delta)$, and two costs associated with the liquidity shocks. The first cost does not depend on the market liquidity parameter λ and represents the opportunity cost of holding cash. This cost goes to zero if the fund has no cash holdings ($\delta = 0$), and goes to a maximum of $\pi\overline{\theta}/2$ in case the fund is perfectly hedged against outflows, which is equal to the expected outflow. The second cost is a complicated function of the market liquidity parameter and represents the expected value of going on the secondary market. In most cases, this term will decrease with the cash quantity, such that adding cash to the portfolio can increase the expected value of the fund by protecting it against liquidity costs on the secondary market. In the end, the fund's trade-off is to balance its liquidity exposure by weighing the opportunity cost of holding cash against a higher probability of survival.

In our framework, the market liquidity parameter λ is central by determining the strength of the trade-off. If λ is high, the asset is equivalent to a liquid short-term asset that will be sold at its virtual fair value on the secondary market. In this case, the opportunity cost of holding cash will dominate Equation (4). On the other hand, if λ is small, the gains of the fund's investment are realized only in the long-run and the hedging value of cash will dominate the fund's trade-off.

3 Benchmark equilibrium results

In the following Section, we solve the fund's portfolio problem for a simple case where $\overline{\theta} = 1$. We provide the resolution of the more general model in the robustness section.

3.1 Solving for the fund's portfolio

We first provide the resulting optimal portfolio from the maximization of Equation (4).

Theorem 1 *When $\overline{\theta} = 1$, the solution to the fund's portfolio problem is available in closed-form and the amount of risky asset is given by:*

$$1 - \delta^* = \frac{1}{\pi} \cdot \frac{(\lambda + 1)\left(1 - \frac{1}{R}\right)}{(\lambda + 1)\left(1 - \frac{1}{R}\right) + 1}. \tag{5}$$

Theorem 1 has an intuitive interpretation. When the funding liquidity risk grows (π is high), the fund decreases its quantity of illiquid asset to increase its chance of survival at the end of the second period. When the secondary market becomes more liquid (λ is high), the fund has a higher demand for the illiquid asset since the opportunity cost of holding cash grows higher. Last and intuitively, when the return of the illiquid asset grows, the fund demands more of it. Note that even if the return of the risky asset was infinite, the fund's portfolio would not be fully loaded with it. Indeed, pushing R to infinity, Equation (5) produces illiquid holdings of $\frac{\lambda+1}{\pi(\lambda+2)}$, which can be as low as $1/2$ for the worst funding and market liquidity conditions. The hedging motive due to the possibility of the fund's failure is a strong driver of the fund's portfolio.

In the following Corollary, we introduce a specific knife-edge case where further simplifications can be obtained.

Corollary 1.1 *If we assume that illiquid asset returns compensate illiquidity risk such that $R = 1 + \frac{1}{\lambda}$, optimal illiquid asset holdings are given by $1 - \delta^* = \frac{1}{2\pi}$ and only move with the funding liquidity risk.*

In Corrolary 1.1, the illiquid asset raw returns are the inverse of the expected haircut that the fund suffers on the secondary market. In this case, any worsening of market liquidity conditions will at the same time increase the attractiveness of the illiquid asset but decrease the survival probability at the fund of a similar amount. The fund thus becomes only sensitive to funding liquidity risk π to form its portfolio.

In the end, we find that funds' cash holdings largely depend on liquidity conditions in predictable and intuitive directions. More risk provides more hedging motives to the manager.

3.2 Probability of liquidation and performance

What are the effects of such liquidity provisions on the fund's survival rate, and on its performance? We provide a theorem to answer these questions in our benchmark framework.

Theorem 2 *When $\overline{\theta} = 1$, the liquidation probability when the fund holds the optimal cash amount δ^* is given by:*

$$Pr\left(F=0\right) = \frac{1-\frac{1}{R}}{\left(1-\frac{1}{R}\right)\left(\lambda+1\right)+1}, \tag{6}$$

and the expected fund value is equal to:

$$\mathbb{E}(F) = 1 - \frac{\pi}{2} + \frac{1}{2}\cdot\frac{1}{\pi}\cdot\frac{\left(\lambda+1\right)\left(1-\frac{1}{R}\right)}{\left(\lambda+1\right)\left(1-\frac{1}{R}\right)+1}(R-1) \tag{7}$$

Theorem 2 allows us to further explore the effects on market and funding liquidity parameters on two major characteristics. First, the larger the return of the illiquid asset R, the bigger the liquidation probability of the fund since its attractiveness pushes the fund to favor it in its portfolio, thus exposing itself to larger market liquidity risk. Second, we see that the liquidation probability of the fund is a decreasing function of λ, that is the more liquid the secondary market, the higher the survival chance of the fund. In turn, the cash holdings of the fund are determined such that its survival probability is insensitive to the funding liquidity parameter π. Thus, when cash holdings are endogenous, the survival probabilities of the funds reflect primarily the heterogeneity of liquidity on the asset side.

We turn now to Equation (7) that summarizes the risk-return relationship of the fund. Combining the result with Theorem 1, we recognize that the fund's expected value can be written $R - \frac{\pi}{2} - \frac{1+\delta^*}{2}(R-1)$. R is the value achieved if there was no liquidity shocks to a fund invested fully in illiquid asset. The second term represents the average outflow of investors at the first period, while the third term summarizes the expected opportunity cost of holding cash. This formulation also allows us to easily obtain the influence of the liquidity parameters on the fund's performance. First, an increase in the average market liquidity λ will produce a higher expected fund performance, since illiquid asset holdings go up and the expected return goes up by more than the liquidation probability. Second, any increase in funding liquidity risk π automatically results in lower expected performance for the fund. This mainly goes through two channels: *(i)* expected client's withdrawal becomes larger, and *(ii)* cash holdings become larger. As we have seen above, π has no impact on the liquidation probability of the fund at equilibrium such that the increase in cash holdings only reduces the expected performance.

In the end, both types of liquidity risk go against the traditional risk-return relationship, where one would expected that more exposure to either funding or market liquidity would produce a compensation for the fund. Our corollary below explores the case where returns of the illiquid asset can compensate the fund for the relative illiquidity of the secondary market.

Corollary 2.1 *If we assume that illiquid asset returns compensate illiquidity risk such that $R = 1 + \frac{1}{\lambda}$, the liquidation probability and expected performance of the fund are given by:*

$$Pr\,(F = 0) = \frac{1}{2(\lambda+1)} \quad \textit{and} \quad \mathbb{E}(F) = 1 - \frac{\pi}{2} + \frac{1}{4\pi\lambda}\,. \tag{8}$$

Corollary 2.1 shows that the expected performance of the fund can decrease with the market liquidity when the investors are compensated for an illiquid risk premia in their risky asset holdings. In the end, the risk-return relationship of the fund's performance is determined by how the market compensates the exposure to higher illiquidity risk.

3.3 Risk compensation and comparative statics

As emphasized previously, the risk-return relationship of the fund's performance can be different depending on the premium of the market. In the following, we consider that the return of the illiquid asset is given by $R = 1 + \frac{\kappa}{\lambda+1-\kappa}$. In this expression, κ can be interpreted as a risk premium parameter that transforms our expressions slightly.

Theorem 3 *When $\overline{\theta} = 1$, and the return of the illiquid asset is given by $R = 1 + \frac{\kappa}{\lambda+1-\kappa}$, the equilibrium quantities are given by:*

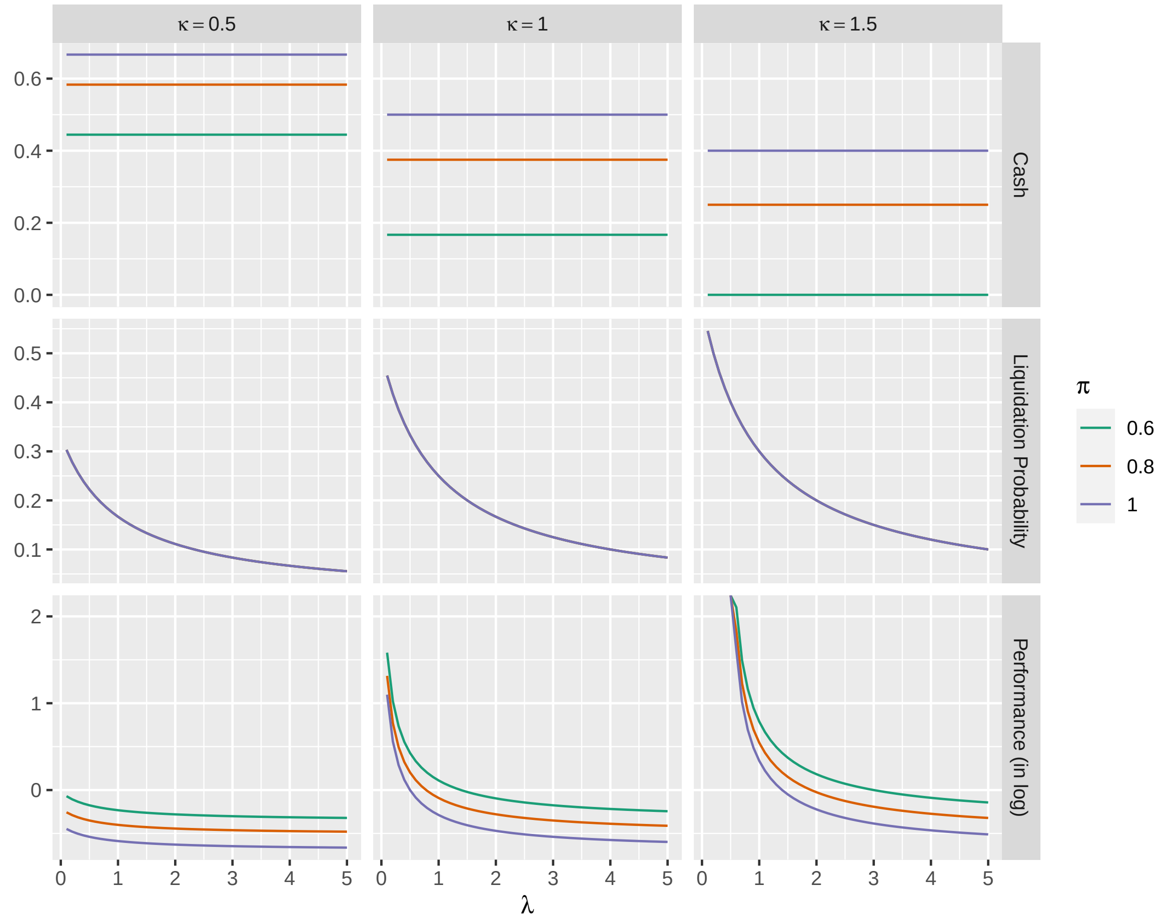
$$1-\delta^* = \frac{1}{\pi}\cdot\frac{\kappa}{\kappa+1}\,, \quad Pr(F=0) = \frac{\kappa}{(\kappa+1)(\lambda+1)}\,, \quad \mathbb{E}(F) = 1 - \frac{\pi}{2} + \frac{\kappa^2}{2\pi(\kappa+1)\,(\lambda+1-\kappa)}\,. \tag{9}$$

The results of the previous theorem are modified in straightforward ways. The larger the risk premium parameter κ, the more attractive the risky asset is such that, *(i)* the larger the illiquid asset holdings, *(ii)* the larger the liquidation probability, and *(iii)* the larger the expected performance of the fund. Unambiguously, we also obtain expected performance as a decreasing function of the market liquidity parameter λ, which restores the natural risk-return relationship.

We perform comparative statics of these different cases, considering several values of (λ, π, κ), for our three quantities of interest. We consider λ varying from 0.1 to 5, corresponding to an average haircut on the secondary market of 90% to close to 15%. For the risk premium parameter, we consider $\kappa \in \{0.5, 1, 1.5\}$, and $\pi \in \{0.6, 0.8, 1\}$.

The results presented on Figure 2 confirm our previous results, and we can additionally gather order of magnitudes from the plot. The optimal liquidity management is quite trivial as it does not depend on the market liquidity parameter λ. The cash quantities shows only large variations with respect to the funding liquidity parameter π a the risk premium parameter κ. For instance, for our baseline case where $\kappa = 1$, the optimal cash quantity goes from 50% when the fund surely experiences an outflow ($\pi = 1$) to below

Figure 2: Comparative statics: baseline



Notes: λ is the degree of liquidity of the secondary market, π is the probability that a non-zero outflow is observed, and κ is a risk premium parameter.

20% when $\pi = 0.6$. This shows that the fund's portfolio is highly sensitive to the funding liquidity risk and to its client's structure. In turn, its survival is only sensitive to the market liquidity parameter[5] λ. Looking at the second row of Figure 2, we see that for our baseline case, the liquidation probability decreases from 45% for a highly illiquid market to about 10% for a highly liquid market. Expected returns are impacted by π all the time and λ only when the illiquidity premium is large. Otherwise λ has no impact on performance. When κ is small, it is essentially π which will move fund returns. Our key message here is thus that the heterogeneity of funds asset and liability sides can be observed through its liquidation probability and cash holdings, respectively.

[5] As the PD depends on λ, we may obtain information on λ from the PD using an implicit approach.

4 Extensions

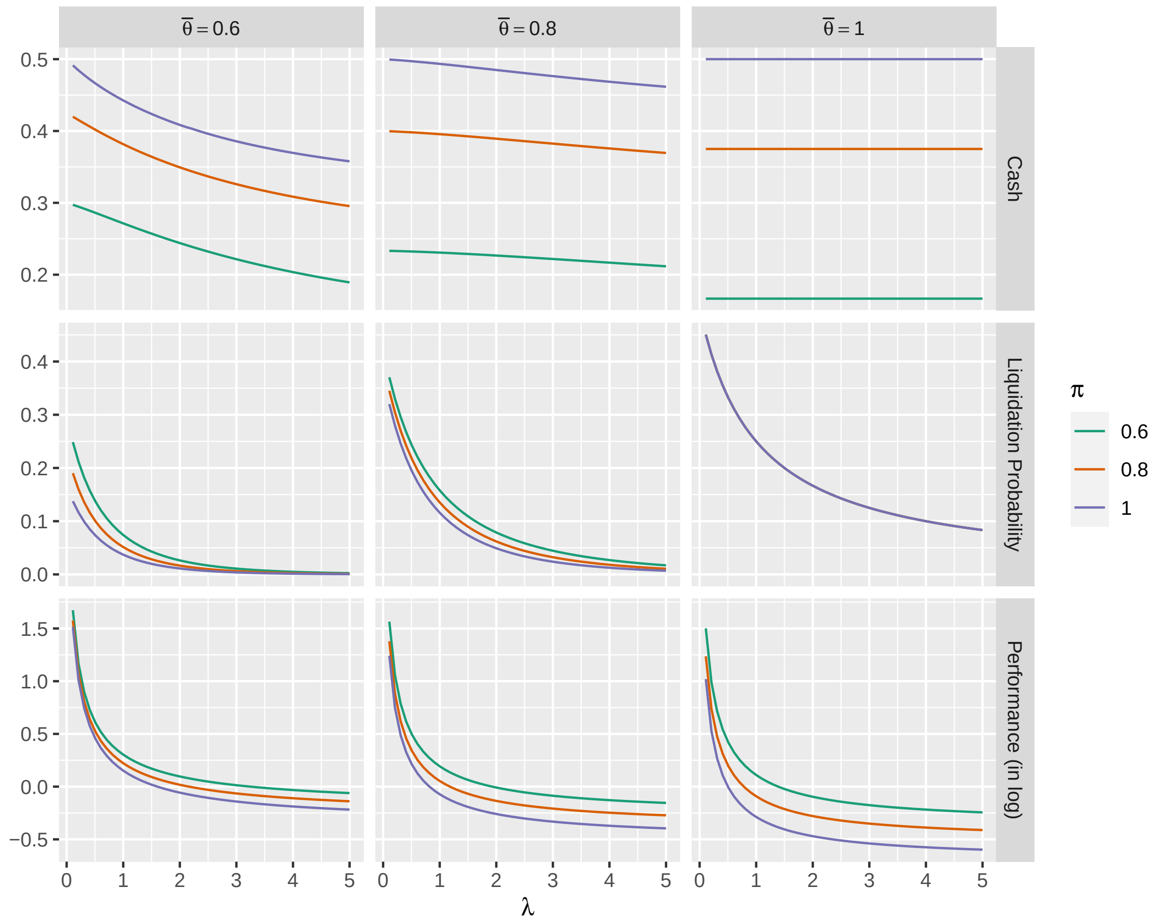
In this section, we extend our previous results by looking at cases where Gates exist ($\overline{\theta} < 1$), and where the fund can have a market impact by selling its assets on the secondary market. We show that the fund's expected performance is still available in closed-form, but the folution to the fund's problem is not. All results are thus computed with numerical optimization in this section.

4.1 The effect of Gates

In the previous section, we only considered the case where the fund did not implement any Gate, i.e. $\overline{\theta} = 1$. We relax this assumption in this Section and explore its effects on our three quantities of interest. To simplify exposition and comparison, we consider hereby that $R = 1 + 1/\lambda$, i.e. our baseline risk premium case. We start back from Equation (7) and solve the maximization program numerically. The results are presented on Figure 3. Gates are expected to decrease liquidity monotonically. But putting gates will have non-trivial effects and especially not monotonous ones. Liquidity management becomes more complex. The optimal cash amount will depend on both π and λ (decreasing in λ), and both dimensions of illiquidity are managed. The optimal cash increases when π is small ("quasi-closed" funds), which is counterintuitive), and decreases when π is high ("quasi-open" funds), which is intuitive. The gate policy is suitable for funds that offer a lot of liquidity to their clients, not for others.

Most of the qualitative results from our benchmark case still hold when we move $\overline{\theta}$. First, the fund's expected performance decreases with $\overline{\theta}$. This result is natural since in our model, the gates are determined exogenously and gather no cost for the fund. Thus, gates protect the fund against funding liquidity shocks and its expected performance increases. As a result, liquidation probabilities are virtually split in half when $\overline{\theta} = 0.6$ compared to $\overline{\theta} = 1$. In addition, the probabilities now vary with the funding liquidity parameter π, even if the lines stay close to each other (see middle row of Figure 3). Interestingly, the liquidation probability grows when the probability of outflow decreases. Indeed, when the fund faces less funding liquidity risk, it shifts its portfolio towards the illiquid asset. Since the illiquid asset is attractive, it is willing to suffer a higher liquidation probability to capture the excess returns provided by its risky investment. Last, the relationship between cash holdings and gates is ambiguous. Comparing $\overline{\theta} = 0.8$ with $\overline{\theta} = 1$ for instance, we see cash holdings can increase or decreases depending on the funding liquidity parameter π. When the latter is equal to 1, cash holdings will increase with $\overline{\theta}$ in order to hedge the additional liability risk. When π is low, cash holdings decrease with $\overline{\theta}$ because any additional hedge from the fund is not worth the opportunity cost of cash. When $\pi = 0.8$,

Figure 3: Comparative statics: gates



Notes: λ is the degree of liquidity of the secondary market, π is the probability that a non-zero outflow is observed, and $\overline{\theta}$ is the Gate parameter. We fix $\kappa = 1$.

we see that cash holdings can increase or decrease with $\overline{\theta}$ depending on the secondary market liquidity. When market liquidity is high, cash holdings decrease with $\overline{\theta}$ because the return of the illiquid asset is not large enough to compensate the additional risk, and vice versa.

In the end, we conclude that including gates in the analysis conserve most of our qualitative findings but complexify the relationship between cash holdings and liquidity risks. Any variation of gates has to be weighted with respect to the existing risk-return relationship. When funding liquidity risk is already high, including gates allows the fund to shift its portfolio to more risky assets. Conversely, when funding liquidity risk is low, including gates is not so important such that the hedging motive dominates. The reasoning follows the same pattern for market liquidity risk.

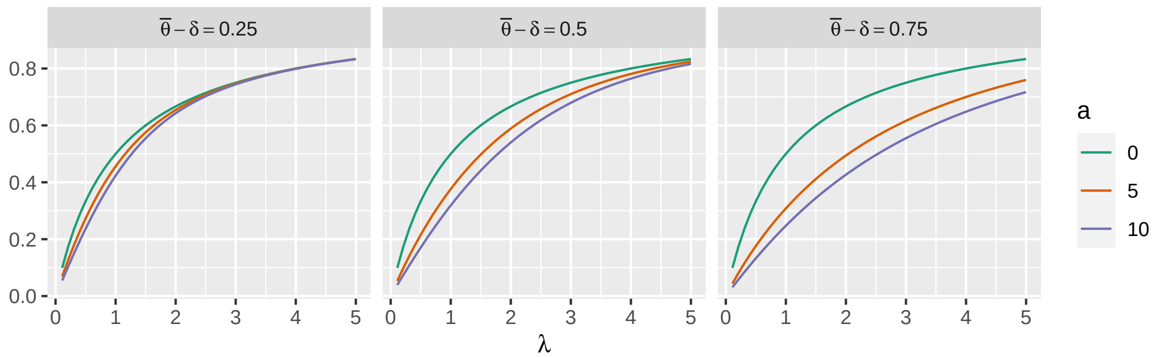
4.2 Spillover effects

Our framework allows us to include spillover effects from funding to market liquidity while conserving the analytical properties of the expected fund performance. We propose a mechanism to correlated the funding and market liquidity shocks aiming at reproducing firesale effects, thus extending the baseline model of Liu and Mello (2011). More specifically, we assume in this section that the selling price on the secondary market is given by:

$$\widetilde{\alpha} = \alpha \cdot \frac{1}{1 + a(\theta - \delta)^{\lambda+1} \cdot \mathbb{1}_{\{\theta > \delta\}}}. \tag{10}$$

While the specification (10) is mainly driven by computational purposes, its economic interpretation is quite simple. First, notice that the specific case $a = 0$ boils down to our benchmark framework presented in the previous Section. Second, we see that the implied haircut on the secondary market grows larger with the unhedged outflow $\theta - \delta$, decreases with the market liquidity λ, and is amplified by the new spillover parameter a. For an easier interpretation of the added parameter, we present the expected selling price of the illiquid asset for different values of a. We distinguish three cases of maximum underfunding: $\overline{\theta} - \delta = \{0.25, 0.5, 0.75\}$ for illustration.

Figure 4: Expected selling price



Notes: λ is the degree of liquidity of the secondary market, a is the spillover parameter and $\overline{\theta}$ is the maximum outflow. δ is fixed and not endogenous in this plot.

The results presented on Figure 4 clearly show the effect of a larger spillover parameter a, and a larger expected underfunding $\overline{\theta} - \delta$. For each plot, keeping δ constant, the expected selling price decreases with the strength of the spillover, and with the maximum underfunding. For instance, for a market applying a 20% haircut without spillover in case the fund needs 50% funding, ($\lambda = 4$, central panel), adding the spillover parameter to 5 and 10 shifts the expected selling price to 78% and 76% respectively. This gap grows with the illiquidity of the market (λ low) and with the expected underfunding ratio.

The previous interpretation did not factor in the endogenous choice of the cash quantity δ. The following theorem proposes a generalization of our results to the case where the fund can suffer from spillovers.

Theorem 4 *Let us define* $\theta^* = \max\left\{x \in [0, \overline{\theta} - \delta] \quad \text{such that} \quad x(1 + ax^{\lambda+1}) \leqslant (1-\delta)\right\}$. *The liquidation probability is given by:*

$$Pr(F=0) = \frac{\pi}{\overline{\theta}} \cdot \left[\overline{\theta} - \delta - \theta^* + \frac{\left(1 + a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(1-\delta)^{\lambda}(\lambda+1)^2}\right]$$

and expected fund performance:

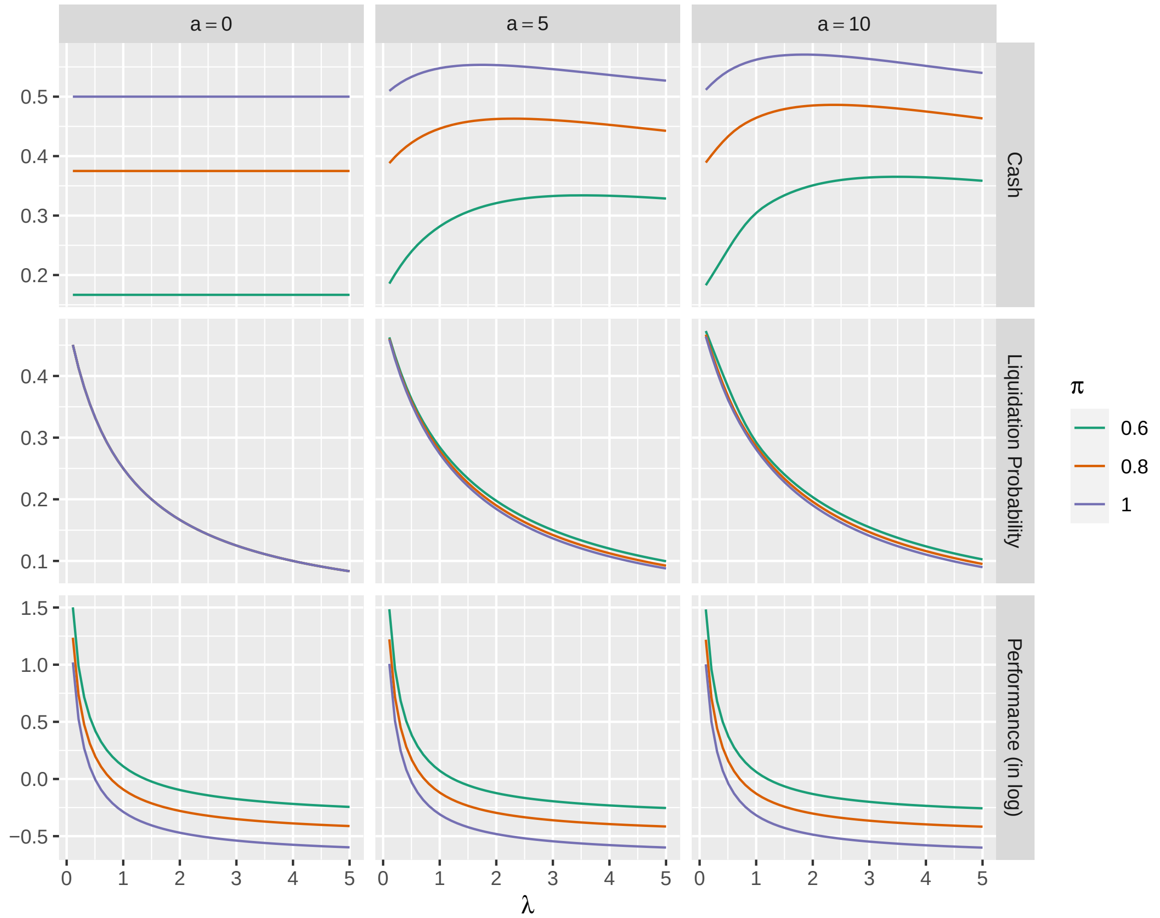
$$\mathbb{E}(F) = (1-\pi)\, F_0 + \pi \cdot \frac{\delta}{\overline{\theta}} \cdot \left(F_0 - \frac{\delta}{2}\right)$$
$$+ \quad \frac{\pi}{\overline{\theta}} \cdot (1-\delta) \cdot R \cdot \left[\theta^* + \frac{\lambda}{(1-\delta)(1-\lambda)}\left(\frac{{\theta^*}^2}{2} + a\frac{{\theta^*}^{\lambda+3}}{\lambda+3}\right) - \frac{\left(1 + a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(1-\delta)^{\lambda}(\lambda+1)^2} \cdot \frac{1}{1-\lambda}\right]$$

When there is no fire sale effect, $a = 0$ and $\theta^* = \overline{\theta} - \delta$, and we obtain the standard case presented above. For any positive a however, we have to compute the result of both the value of θ^* and the maximum expected performance numerically. We present the results on Figure 5. In the left column, we put back the results without spillovers for the sake of comparison. Liquidity management becomes even more complex. Optimal cash policy becomes very dependent on λ with non-linear effects.

When the spillover parameter grows, the fund is leaning more towards cash to hedge for the extra market liquidity risk. In other words, its trying to mitigate risk by reducing the amount of secondary asset it needs to sell if clients ask to get out. The effect can be quite large, especially when the outflow probability is at low ($\pi = 0.6$, green lines). Indeed, for a low outflow probability, the illiquid asset is very attractive and the fund only needs to keep less than 20% of cash on its balance sheet, whatever the market liquidity. In turn, with spillover effects, if a funding liquidity shock happens, a low level of cash holding gets a lot worse for the fund since its haircut has a large extra component that reduces the expected attractiveness of the illiquid asset. The attractiveness of the illiquid asset thus becomes inherently linked to the cash holdings of the fund, changing the risk return trade-off of the fund. Last, Figure 5 shows that adding spillovers from funding to market liquidity risk increases the liquidation probability, and reduces the fund's expected performance, but these effects are rather small in magnitude.

In summary, adding spillover effects does not change the qualitative features of our findings. The only significant deviation is the relationship between cash and liquidity risk, where cash holdings can now vary differently with respect to market liquidity, depending

Figure 5: Comparative statics: spillover



Notes: λ is the degree of liquidity of the secondary market, π is the probability that a non-zero outflow is observed, and a is the spillover parameter. We fix $\kappa = 1$, $\overline{\theta} = 1$.

on the expected spillover.

5 Empirical Application

In this section, we calibrate the model to real hedge fund data to obtain some of the industry characteristics as time series. More precisely, we first use some time series of statistics derived from the cross-section of funds' observed liquidations to filter at each date the illiquid asset returns as well as the portfolio liquidity characteristics, i.e. the effective allocation between liquid and illiquid assets. In a second step, we take the market characteristics as given and perform counterfactual experiments, varying gates, or spillover effects.

5.1 The data

The Lipper TASS database consists of monthly returns, Asset Under Management (AUM) and other HF characteristics for individual funds from May 1973 to October 2015.[6] The database categorizes HF into "Live" and "Graveyard" funds. We apply a series of filters to the data. First, we select only funds with Net Asset Value (NAV) written in USD, with monthly reporting frequency. This avoids double counting, since the same fund can have shares written in USD and Euro for example.

[6]Tremont Advisory Shareholders Services. Further information about this database is provided on the website http://www.lipperweb.com/products/LipperHedgeFundDatabase.aspx.

Table 1: The database.

	Live funds		Liquidated funds		Total	
	(#)	(%)	(#)	(%)	(#)	(%)
Conv	23	2.14	195	3.83	218	3.54
Em	153	14.26	495	9.73	648	10.52
Emn	41	3.82	362	7.12	403	6.54
Ed	101	9.41	511	10.05	612	9.94
Fi	35	3.26	208	4.09	243	3.94
Gm	69	6.43	401	7.82	470	7.63
Lse	407	37.93	1914	37.63	2321	37.68
Mf	134	12.49	573	11.26	707	11.48
Ms	110	10.25	428	8.41	538	8.73
Total	**1073**	**100**	**5087**	**100**	**6160**	**100**

Notes: Codes are as follows: Conv: *convertible arbitrage*, Em: *emerging markets*, Emn: *equity market neutral*, Ed: *event driven*, Fi: *fixed income arbitrage*, Gm: global macro, Lse: *long/short equity hedge*, Mf: *managed futures*, Ms: *multi-strategy*. The table provides the distribution of iive funds on October 2015, and funds liquidated prior to October 2015, across the nine management styles.

We obtain 2353 funds in the "Live" base, and 8826 liquidated funds. Second, we only consider HF reporting their Asset Under Management on a regular basis. This information is essential to compute the time series of inflows and outflows. Third, to keep the interpretation in terms of individual funds, we eliminate the funds of funds and, for funds with multiple share classes, we eliminate duplicate share classes from the sample. Finally, we select the nine management styles with a sufficiently large size. These are Long/Short Equity Hedge (Lse), Event Driven (Ed), Managed Futures (Mf), Equity Market Neutral (Emn), Fixed Income Arbitrage (Fi), Global Macro (Gm), Emerging Markets (Em), Multi Strategy (Ms), and Convertible Arbitrage (Conv). After applying all these filters, we get 1073 funds in the "Live" database and 5087 liquidated funds. The distribution by style of live and liquidated funds in the database is reported in Table 1. The largest management style in the database of live and liquidated funds is Long/Short Equity Hedge (about 40%), followed by Managed Futures, Multi-Strategy and Event Driven (each about 10%). In the following, we only consider in the following the Long/Short Equity Hedge strategy. In addition, to obtain more reliable results, we start the sample in 2000 only where the number of funds is sufficient.

We denote the time index by $t \in \{1, \ldots, T\}$ and the fund index by $i \in \{1, \ldots, N_t\}$, where T is the total number of periods in the sample and N_t is the total number of funds alive at date t. The fund flows $f_{i,t}$ are the growth rate of the assets under management as a percentage of the net asset value minus the performance. The fund default variable $d_{i,t} = 1$ if i^{th} fund net asset value is not reported at t whereas it was reported before.

We compute two baseline series to guide our empirical exercise. The first important input for our model is the probability of experiencing an outflow π. The second important observable is the liquidation probability, which will help us back out the primitives of the fund's optimization problem, such as the average market liquidity and the optimal cash amount. We compute these estimates from the data with simple plug-in estimators, such that:

$$\widehat{\pi}_t = \frac{1}{N_t} \sum_{i=1}^{N_t} \mathbb{1}\{f_{i,t} < 0\} \quad \text{and} \quad \widehat{PL}_t = \frac{1}{N_{t-1}} \sum_{i=1}^{N_{t-1}} d_{i,t} \,. \tag{11}$$

$\widehat{\pi}_t$ is the average across all funds of the number of funds who experience outflows in a given month. $\widehat{PL}_t$ is the average across funds of funds who were alive at $t-1$ and stop reporting at t. These time series are represented on Figure 6, panels (a) and (b), respectively. On panel (a), we see that the average outflow probability is quite high at about 45% across the whole sample. It trends upwards, from 40% to 60% between 2000 and 2015. Importantly, the great financial crisis shows a large spike at 65%, showing that a lot of clients were eager to pull their investments out from the hedge fund industry. Interestingly, this spike is much less pronounced for the liquidation probability of hedge funds (panel b), showing that funds were hedging part of their risk on the liability side. The overall liquidation probability is much smaller, between 0.5% and 4%, where the biggest spike can be observed during the European debt crisis in 2012. These two series together inform us about the funds' trade-off.

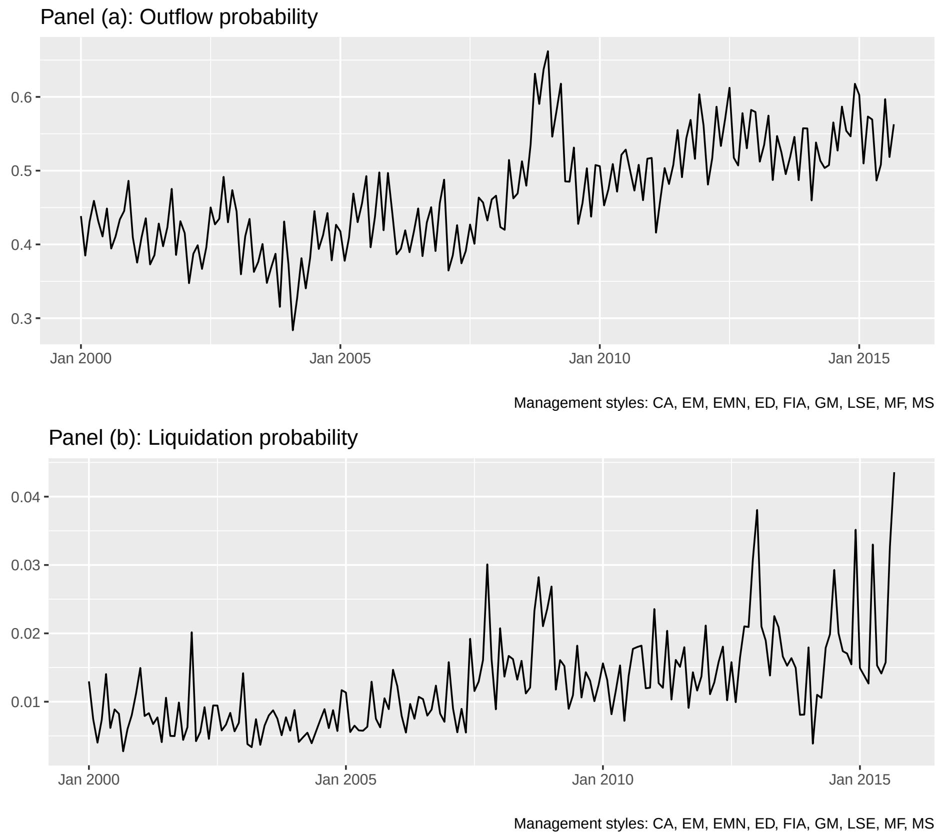
5.2 Calibration method

In our first calibration exercise, we assume no liquidity spillover and set $a = 0$. There are four unknowns to the model: the outflow probability, which we proxy by $\widehat{\pi}$, the market liquidity index λ, the risky asset return R, and the gates parameter $\overline{\theta}$. To simplify our problem, we consider the version of our model developed in Section 3.3, where we set $\kappa = 0.5$. Thus we set:

$$\widehat{R} = 1 + \frac{1}{2\lambda + 1} \,. \tag{12}$$

Second, to provide a more realistic framework, we assume that the gates parameters is set at 0.6, such that only 60% of the liabilities of the fund can be retrieved by investors. This also allows us to get interior solutions to the cash optimization problem given the

Figure 6: Data Series: Probabilities



Notes: This Figure represents the estimated series for the outflow (panel a) and liquidation probability (panel b). These series are computed as averages across funds, as defined by Equation (11). Management styles are Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Fixed Income Arbitrage (FIA), Global Macro (GM), Long/Short Equity Hedge (LSE), Managed Futures (MF), and Multi Strategy (MS). Data is monthly from January 2000 to October 2015.

outflow and liquidation probabilities derived from data.

Our last degree of freedom is the degree of market liquidity λ. We estimate it from data by minimizing the squared distance between the model-implied and data liquidation probability $\widehat{PL}$. Our approach thus allows to back out simultaneously the cash amount and market liquidity from from the data, the latter allowing us to obtain the risky asset return accordingly with Equation (12).

5.3 Estimation results

Our results are presented on Figure 7. On panel (a), we see that the cash amount (or the liquid part in the portfolio allocation) moves hand-in-hand with the probability of outflow presented on Figure 6, panel (a). This means that, accordingly with our theory presented above, the fund decides its cash/liquidity policy primarily to hedge its liquidity risk on the liability side. Indeed, looking back at Theorem 3, we see that $\kappa = 0.5$ would imply a cash policy of $1 - \frac{1}{3\pi}$ if there was no gates. In the presence of gates, the trade-off may be modified slightly but it remains that the optimal cash/liquid amount is increasing in the outflow probability. The optimal cash/liquid amount fluctuates between 0.1 in 2004 and 0.5 during the great financial crisis, that is between 15% and 85% of the immediately pledge-able liabilities. This magnitude is quite high, and can be put in perspective if we interpret the cash/liquid amount as amounts that are available for borrowing for the fund. Indeed, funds have a maximum leverage that is not binding in practice. They can thus raise cash by increasing their effective leverage. We learn form this result that, since 2005, hedge funds have increased their liquid investment to become funds like mutual funds. They provide liquidity by investing in liquid assets.

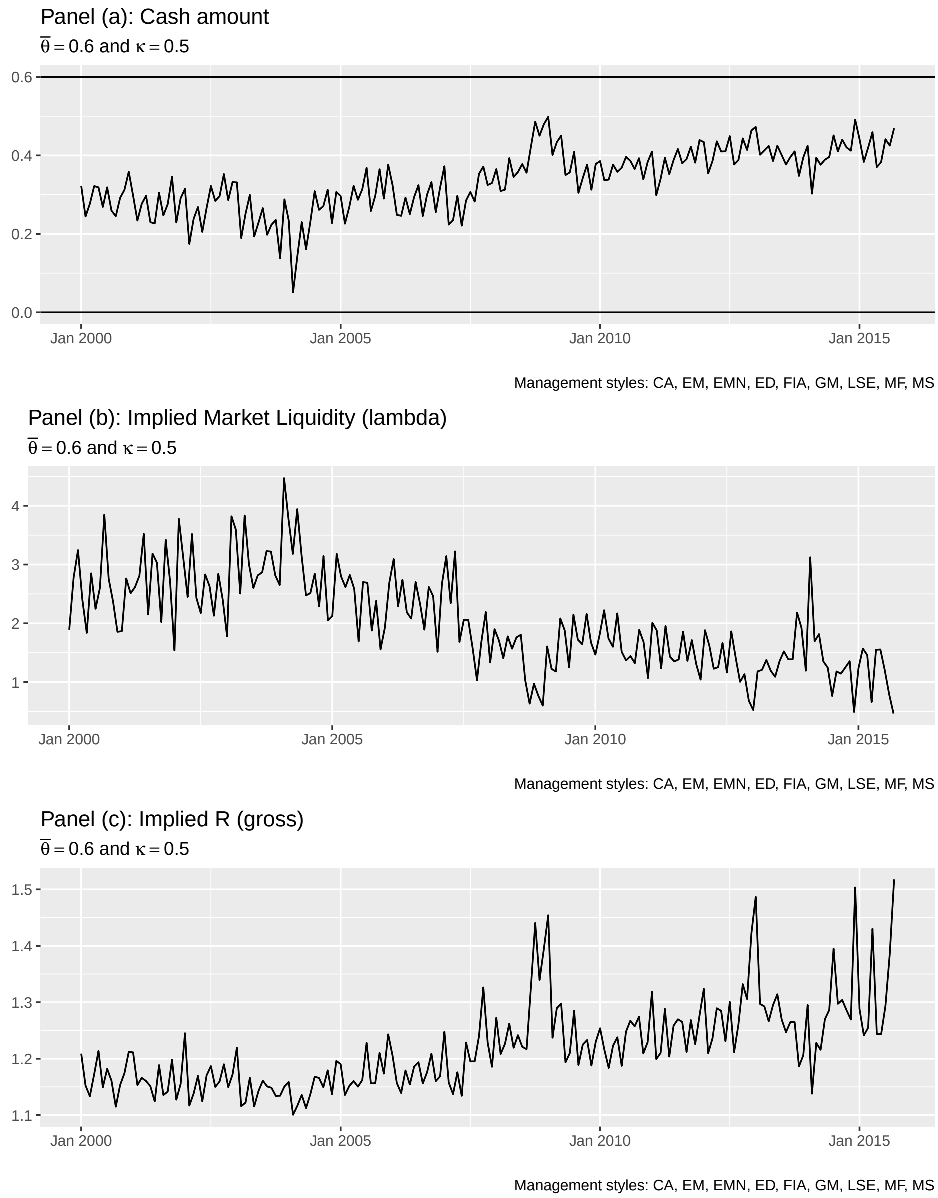
Second, panel (b) of Figure 7 presents the estimated degree of market liquidity consistent with the observed liquidation probability. Comparing with panel (b) of Figure 6, there is an obvious negative correlation between the two series, meaning that most of the increase in the liquidation probability is due to a relatively poorer market liquidity in hedge fund investments over time. Market liquidity hits its lowest at 0.5 during the great financial crisis, the European debt crisis, and towards the end of the sample. The put some economic magnitude on these numbers, remember that the average haircut that is applied on the secondary market for the illiquid asset is equal to $1 - \frac{\lambda}{\lambda+1}$, corresponding to as much as 65% in these periods of poor liquidity. In periods of better liquidity conditions such as the first half of the sample, the average haircut is about 25%, sufficiently to make about 1% of hedge fund go into liquidation. Last, panel (c) presents the implied risky asset return that is inversely related to the degree of market liquidity, by assumption. The return can be quite high in bad liquidity times, representing the fact that the price falls down during crises such that the expected return increases.

5.4 Counterfactual experiments

In this section, we consider some counterfactual scenarios where we make the gates, risk compensation, and spillover components vary, while keeping everything else constant. More specifically, we take as inputs the outflow probabilities $\widehat{\pi}$ and the estimated market liquidity series $\widehat{\lambda}$ presented in the previous sections. We recompute the optimal cash/liquidity policy and the liquidation probability implied by making the other components vary one at a time.

We first focus on the impact of gates, presented on Figure 8. We can answer the question of what would have happened if hedge funds could have limited the amount of liquidity risk on the liability side by lowering the gates to 50% rather than 60%. Unsurprisingly, the hedge fund is able to unload some of its liquid asset and tilt its

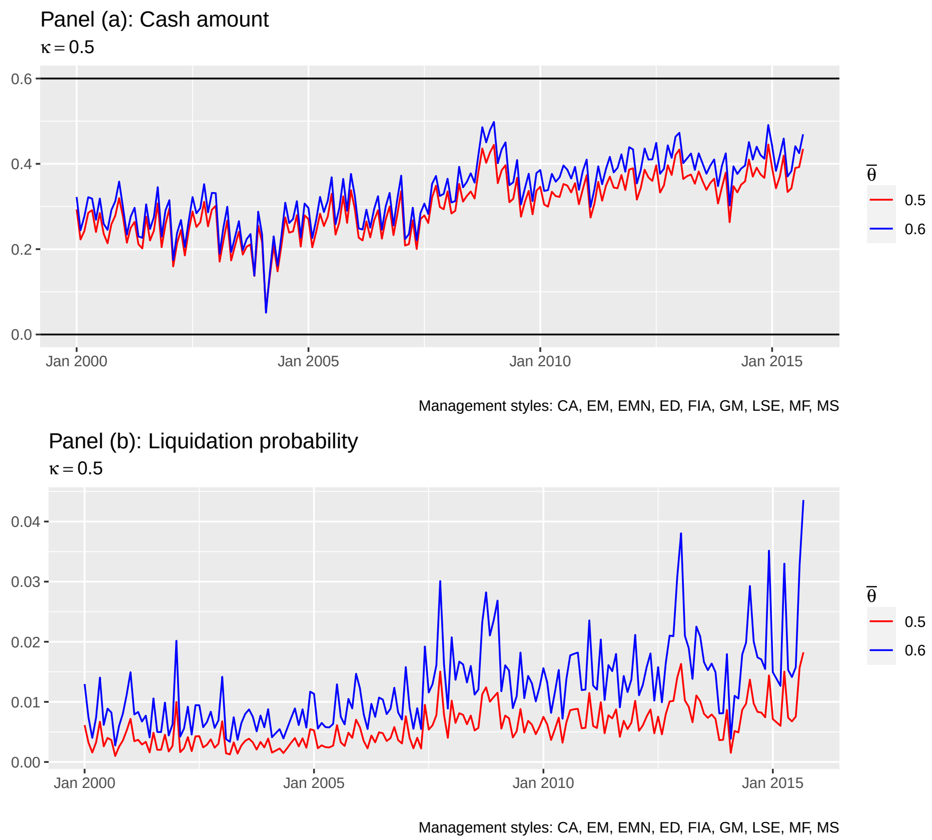
Figure 7: Model Outputs: Cash, Return, and Market Liquidity



Notes: This Figure represents the estimated series for the optimal cash policy δ^* undertaken by the hedge fund (panel a), the degree of market liquidity $\widehat{\lambda}$ backed out from observable data (panel b), and the implied risky return $\widehat{R}$ as defined by Equation (12). Management styles considered to compute the outflow and liquidation probabilities – inputs of the estimation method – are Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Fixed Income Arbitrage (FIA), Global Macro (GM), Long/Short Equity Hedge (LSE), Managed Futures (MF), and Multi Strategy (MS). Data is monthly from January 2000 to October 2015.

portfolio towards the illiquid asset (panel a). The decrease is quite modest nonetheless, of a maximum of 0.05 during the great financial crisis. However, while doing so, it is able to also reduced its liquidation probability and cut it by more than half since a lower amount is directly pledgeable by clients on its liability side. Therefore, gates have the positive impact of protecting the fund while allowing it to capture more of the illiquid asset premium and thus increase its expected return. While, as mentioned before, gates may have other costs that are outside the range of our model, we emphasize that they can be beneficial to the clients as it allows the fund to capture more of the market liquidity premium.

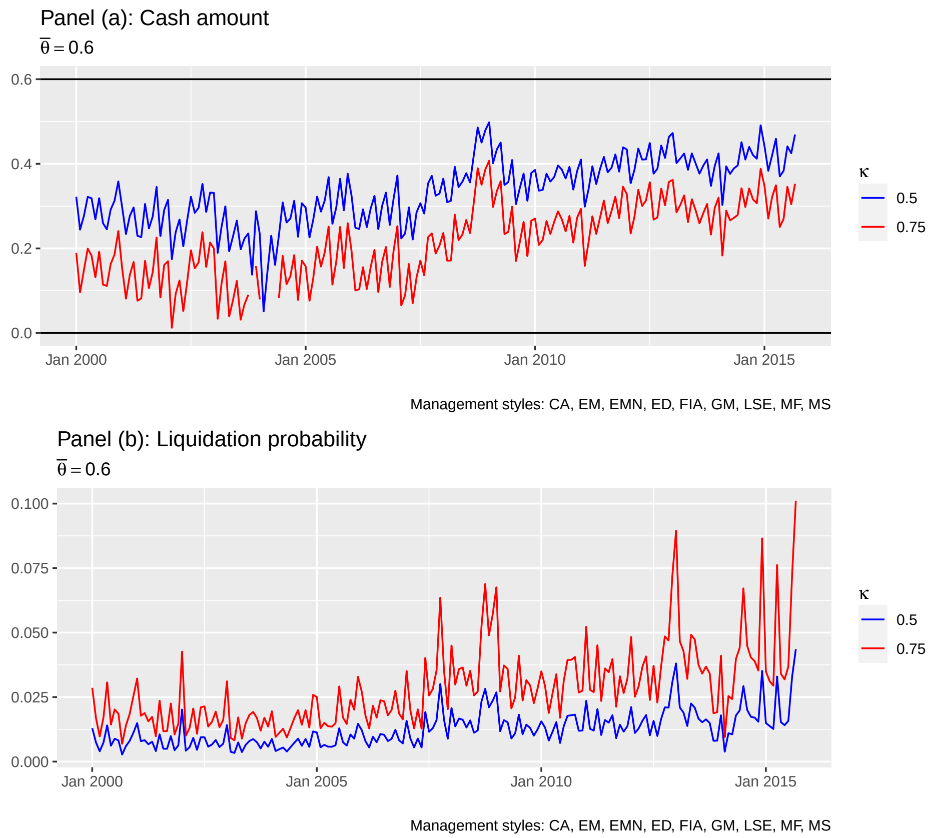
Figure 8: Model Outputs: alternative gates



Notes: This Figure represents the estimated series for the optimal cash policy δ^* undertaken by the hedge fund (panel a), and the liquidation probability (panel b) as implied by our model. Inputs are the outflow probabilities of Figure 6 panel (a), as well as the estimated market liquidity conditions and risky asset returns estimated on Figure 7. Blue curves are the baseline model result with $\overline{\theta} = 0.6$, red curves present the counterfactual results obtained with $\overline{\theta} = 0.5$. Management styles considered to compute the outflow and liquidation probabilities – inputs of the estimation method – are Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Fixed Income Arbitrage (FIA), Global Macro (GM), Long/Short Equity Hedge (LSE), Managed Futures (MF), and Multi Strategy (MS). Data is monthly from January 2000 to October 2015.

Second, we study the effect of alternative risk compensation by switching $\kappa = 0.5$ to $\kappa = 0.75$. In the latter case, the risky return is given by $R = 1 + \frac{3}{4(\lambda+1)-3}$ and increases. Results are presented on Figure 9. On panel (a), we see that the quantity of cash/liquid asset held by the hedge fund decreases as a result of the increased attractiveness of the illiquid asset. The decrease is more pronounced up to 0.1 when the outflow probability is lower, at the beginning of the sample for instance. As a result, panel (b) shows that the liquidation probability increases significantly and even reaches 10% by the end of the sample. The shift in the risk-return trade-off for the illiquid asset leads the hedge to take on more risk to try to capture more of the market liquidity premium.

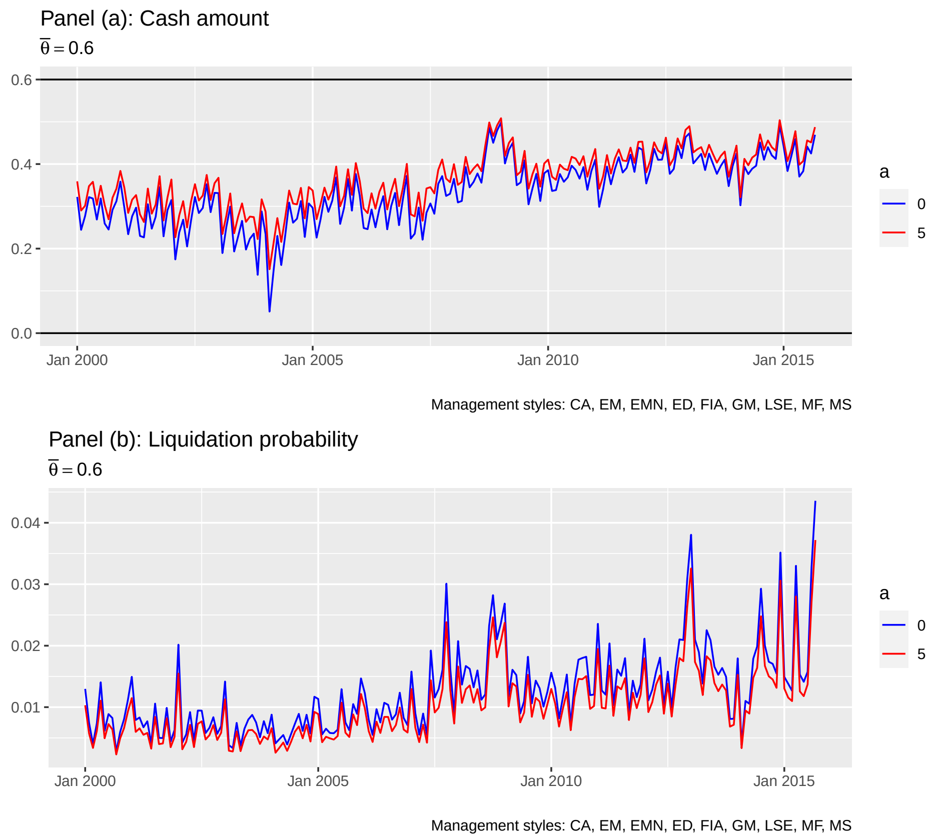
Figure 9: Model Outputs: alternative premia



Notes: This Figure represents the estimated series for the optimal cash policy δ^* undertaken by the hedge fund (panel a), and the liquidation probability (panel b) as implied by our model. Inputs are the outflow probabilities of Figure 6 panel (a), as well as the estimated market liquidity conditions and risky asset returns estimated on Figure 7. Blue curves are the baseline model result with $\kappa = 0.5$, red curves present the counterfactual results obtained with $\kappa = 0.75$. Management styles considered to compute the outflow and liquidation probabilities – inputs of the estimation method – are Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Fixed Income Arbitrage (FIA), Global Macro (GM), Long/Short Equity Hedge (LSE), Managed Futures (MF), and Multi Strategy (MS). Data is monthly from January 2000 to October 2015.

Last, we consider a counterfactual scenario where there are liquidity spillovers ($a \neq 0$) and the amount that the fund liquidates on the secondary market feeds back onto the selling price that it can secure. We switch from a baseline scenario where there is no liquidity spillover to a parameter of $a = 5$. Results are presented on Figure 10 in blue for the baseline and red for the counterfactual scenario. When spillovers are present, the fund endogenizes the risk and holds a larger amount of cash/liquid asset with respect to the baseline scenario. The increase is the largest when market liquidity is less of an issue, such as during 2004 where the amount of cash switches from 0.05 to nearly 0.15. Instead, during the great financial crisis, where the market liquidity is poor, there are nearly no difference in the cash/liquidity buffers. Interestingly, and despite the increase of risk when the fund is liquidating its illiquid asset, the overall impact on the liquidation probability is negative such that the fund is less likely to cease its activity. So liquidity spillovers can lead the fund to take on less risk, both in terms of market liquidity exposure and liquidation probability.

Figure 10: Model Outputs: alternative spillover



Notes: This Figure represents the estimated series for the optimal cash policy δ^* undertaken by the hedge fund (panel a), and the liquidation probability (panel b) as implied by our model. Inputs are the outflow probabilities of Figure 6 panel (a), as well as the estimated market liquidity conditions and risky asset returns estimated on Figure 7. Blue curves are the baseline model result with $a = 0$, red curves present the counterfactual results obtained with $a = 5$. Management styles considered to compute the outflow and liquidation probabilities – inputs of the estimation method – are Convertible Arbitrage (CA), Emerging Markets (EM), Equity Market Neutral (EMN), Event Driven (ED), Fixed Income Arbitrage (FIA), Global Macro (GM), Long/Short Equity Hedge (LSE), Managed Futures (MF), and Multi Strategy (MS). Data is monthly from January 2000 to October 2015.

6 Conclusion

In this paper, we show that hedge funds face liquidity risks both on the asset and liability sides that lead them to keep a positive amount of liquid assets to survive and capture the liquidity premium of illiquid assets. We formalize the fund's trade-off in a two-period model where it has access to a liquid asset and an illiquid asset, and where clients can ask to redeem their shares before the illiquid asset return is realized. After characterizing the optimal liquidity policy, we show that imposing gates in some cases can be beneficial since it allows the fund to take on more risk on the asset side while limiting its risk on the liability side. Our results are unaffected qualitatively by the presence of liquidity spillovers from the liability side to the asset side. Last, we provide an empirical calibration on hedge fund data to show how the different sorts of liquidity risks have evolved during the 2000-2015 sample.

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A Appendix

A.1 Case by case fund value

In the case of no liquidity shock, the fund collects the proceeds of its investment, i.e. cash δ and the return from the risky asset $(1-\delta)R$. We denote by:

$$F_0 = \delta + (1-\delta)R\,.$$

In case of a small liquidity shock, this quantity is slashed from the cash that is getting out, i.e. $F_0 - \theta$. In case of a large liquidity shock, obtaining Equation (2) takes a bit more work. The fund walks out with only the proceeds of what remains of its risky asset investment. Since it liquidates a proportion γ of its risky position, we have:

$$\begin{aligned} F_1 &= (1-\gamma)(1-\delta)R \\ &= \delta + (1-\delta)R + (1-\gamma)(1-\delta)R - \delta - (1-\delta)R \\ &= F_0 - \gamma(1-\delta)R - \delta\,. \end{aligned}$$

Then, since the fund liquidates only the necessary amount, we have $\gamma(1-\delta)\alpha = \theta - \delta$. We then introduce θ as:

$$\begin{aligned} F_1 &= F_0 - \gamma(1-\delta)R - \delta - \theta + \delta + \gamma(1-\delta)\alpha \\ &= F_0 - \theta - \gamma(1-\delta)(R-\alpha)\,.\blacksquare \end{aligned}$$

A.2 Solving the model

To solve it analytically, we express its expectation from period $t = 0$ using the law of iterated expectations:

$$\begin{aligned} \mathbb{E}(F) &= (1-\pi)\,\mathbb{E}(F_0) + \pi \cdot \mathbb{P}(\theta < \delta) \cdot \mathbb{E}(F_0 - \theta \,|\, \theta < \delta) \\ &+ \pi \cdot \mathbb{P}(\theta > \delta) \cdot \mathbb{P}(\gamma \leqslant 1 \,|\, \theta > \delta) \cdot \mathbb{E}\left[F_0 - \theta - \gamma(1-\delta)(R-\alpha) \,|\, \theta > \delta, \gamma \leqslant 1\right]\,. \end{aligned}$$

The probability appearing on the second row is the joint probability of experiencing a funding liquidity shock which is big enough to trigger liquidation of the risky asset, and that the haircut on the secondary market is small enough to be alive at the last period.

We can simplify as:

$$\begin{aligned}\mathbb{E}(F) &= (1-\pi)\,F_0 + \pi\cdot\frac{\delta}{\overline{\theta}}\cdot\left(F_0-\frac{\delta}{2}\right)\\ &+ \quad \pi\cdot\frac{\overline{\theta}-\delta}{\overline{\theta}}\cdot(1-\delta)\cdot R\cdot\mathbb{P}(\gamma\leqslant 1\,|\,\theta>\delta)\cdot\mathbb{E}\left[(1-\gamma)\,|\,\theta>\delta,\gamma\leqslant 1\right].\end{aligned}$$

The main difficulty resides in calculating the last row of the previous Equation. It is useful to simplify it as follows:

$$\mathbb{P}(\gamma\leqslant 1\,|\,\theta>\delta)\cdot\mathbb{E}\left[1-\gamma\,|\,\theta>\delta,\gamma\leqslant 1\right]=\mathbb{P}(\gamma\leqslant 1|\theta>\delta)-\mathbb{E}\left(\gamma\mathbb{1}_{\{\gamma\leqslant 1\}}|\theta>\delta\right). \tag{13}$$

The proof is easily obtained using the law of total probability.[7] The computation of these two unknown terms are detailed in the next section.

A.2.1 Liquidation probability given the cash holding δ

We prove all results in the general case of Section 4.2 and derive the specific cases in the following sections.

Theorem 5 *Given a cash holding level δ and the occurrence of a big liquidity shock ($\theta>\delta$), the conditional liquidation probability is given by:*

$$\mathbb{P}\left(\gamma>1|\theta>\delta\right)=1-\frac{\theta^*}{\overline{\theta}-\delta}+\frac{\left(1+a{\theta^*}^{\lambda+1}\right)^{\lambda+1}-1}{a(\overline{\theta}-\delta)(1-\delta)^{\lambda}(\lambda+1)^2}, \tag{14}$$

$$\textit{where}\qquad \theta^*=\max\left\{x\in[0,\overline{\theta}-\delta]\quad\textit{such that}\quad x(1+ax^{\lambda+1})\leqslant 1-\delta\right\}. \tag{15}$$

Note that when $a=0$, $\theta^*=\overline{\theta}-\delta$. Conversely, when $a\to+\infty$, $\theta^*\to 0$. When $a=0$, we have:

$$\frac{\left(1+a{\theta^*}^{\lambda+1}\right)^{\lambda+1}-1}{a(\overline{\theta}-\delta)(1-\delta)^{\lambda}(\lambda+1)^2}\sim\frac{\left(\overline{\theta}-\delta\right)^{\lambda}}{(\lambda+1)\left(1-\delta\right)^{\lambda}} \tag{16}$$

and the conditional liquidation probability is given by:

$$\mathbb{P}\left(\gamma>1|\theta>\delta\right)=\frac{1}{\lambda+1}\cdot\left(\frac{\overline{\theta}-\delta}{1-\delta}\right)^{\lambda}$$

When in addition $\overline{\theta}=1$, the conditional liquidation probability is independent from δ and is given by:

$$\mathbb{P}\left(\gamma>1|\theta>\delta\right)=\frac{1}{\lambda+1}.$$

[7] $\mathbb{E}\left(\gamma\mathbb{1}_{\{\gamma\leqslant 1\}}\right)=\mathbb{P}(\gamma\leqslant 1)\mathbb{E}\left(\gamma\mathbb{1}_{\{\gamma\leqslant 1\}}|\gamma\leqslant 1\right)+\mathbb{P}(\gamma\geqslant 1)\mathbb{E}\left(\gamma\mathbb{1}_{\{\gamma\leqslant 1\}}|\gamma\geqslant 1\right)=\mathbb{P}(\gamma\leqslant 1)\mathbb{E}\left(\gamma|\gamma\leqslant 1\right)$.

Corollary 5.1 *Given a cash holding level δ, the fund liquidation probability is given by:*

$$\mathbb{P}(\gamma > 1) = \frac{\pi}{\overline{\theta}} \left[\overline{\theta} - \delta - \theta^* + \frac{\left(1 + a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(1-\delta)^\lambda(\lambda+1)^2} \right]. \tag{17}$$

When $a = 0$, this probability simplifies to:

$$\mathbb{P}(\gamma > 1) = \frac{\pi}{\overline{\theta}} \cdot \frac{(\overline{\theta} - \delta)^{\lambda+1}}{(\lambda+1)(1-\delta)^\lambda}.$$

For the sake of notational simplicity, we first define $\widetilde{\theta} = \theta - \delta$. Conditionally on the combination of a liquidity shock happening and that the liquidation of the risky asset must be performed, we have:

$$\widetilde{\theta} | \theta > \delta \sim \mathcal{U}(0, \overline{\theta} - \delta),$$

that is the conditional pdf of $\widetilde{\theta}$ is given by $f_{\widetilde{\theta}}(x) = \frac{\mathbb{1}_{\{x \in [0, \overline{\theta} - \delta]\}}}{\overline{\theta} - \delta}$. Remember that the conditional pdf of $-\log \alpha$ is given by $f(x) = \mathbb{1}_{\{x \geqslant 0\}} \lambda \exp(-\lambda x)$.

The default probability is the probability that the proportion of risky asset to liquidate exceeds 100%. We therefore compute:

$$\begin{aligned}
\mathbb{P}\left(\gamma > 1 | \widetilde{\theta} > 0\right) &= \mathbb{P}\left[\frac{\widetilde{\theta}\left(1 + a\widetilde{\theta}^{\lambda+1}\right)}{1-\delta} \cdot \frac{1}{\alpha} > 1 | \widetilde{\theta} > 0 \right] \\
&= \mathbb{E}\left\{ \mathbb{E}\left[\mathbb{1}\left\{ \frac{\widetilde{\theta}\left(1 + a\widetilde{\theta}^{\lambda+1}\right)}{1-\delta} \cdot \frac{1}{\alpha} > 1 \right\} |\widetilde{\theta} \right] |\widetilde{\theta} > 0 \right\} \\
&= \mathbb{E}\left\{ \mathbb{P}\left[\frac{1}{\alpha} > \frac{1-\delta}{\widetilde{\theta}\left(1 + a\widetilde{\theta}^{\lambda+1}\right)} |\widetilde{\theta} \right] |\widetilde{\theta} > 0 \right\}
\end{aligned}$$

Two cases must be distinguished. It is possible that the ratio inside the probability is lower than 1. In this case, the conditional probability is exactly equal to 1 since α has support on $(0, 1)$. We can then use the fact that $\alpha = e^{-\beta}$ where $\beta \sim \mathcal{E}xp(\lambda)$ and the properties of the exponential distribution to write the following simplification:

$$\begin{aligned}
\mathbb{P}\left(\gamma > 1 | \widetilde{\theta} > 0\right) &= \mathbb{E}\left[\mathbb{1}\left\{ \frac{1-\delta}{\widetilde{\theta}\left(1 + a\widetilde{\theta}^{\lambda+1}\right)} > 1 \right\} \left(\frac{\widetilde{\theta}^\lambda \left(1 + a\widetilde{\theta}^{\lambda+1}\right)^\lambda}{(1-\delta)^\lambda} - 1 \right) + 1 \middle| \widetilde{\theta} > 0 \right] \\
&= 1 + \int_0^{\overline{\theta} - \delta} \mathbb{1}\left\{ x\left(1 + a x^{\lambda+1}\right) < 1 - \delta \right\} \left[\frac{x^\lambda \left(1 + a x^{\lambda+1}\right)^\lambda}{(1-\delta)^\lambda} - 1 \right] \frac{1}{\overline{\theta} - \delta} \, \mathrm{d}x
\end{aligned}$$

We define:

$$\theta^* = \max\left\{x \in [0, \overline{\theta} - \delta] \quad \text{such that} \quad x(1 + ax^{\lambda+1}) \leqslant 1 - \delta\right\}\,.$$

Noting that:

$$\int x^{\lambda}\left(1 + ax^{\lambda+1}\right)^{\lambda} = \frac{\left(1 + ax^{\lambda+1}\right)^{\lambda+1}}{a(\lambda+1)^2}$$

The integral now writes:

$$\begin{aligned}
\mathbb{P}\left(\gamma > 1|\widetilde{\theta} > 0\right) &= 1 + \int_0^{\theta^*} \frac{x^{\lambda}\left(1 + ax^{\lambda+1}\right)^{\lambda}}{(1-\delta)^{\lambda}} \frac{1}{\overline{\theta} - \delta}\,\mathrm{d}x - \frac{\theta^*}{\overline{\theta} - \delta} \\
&= 1 - \frac{\theta^*}{\overline{\theta} - \delta} + \frac{1}{(1-\delta)^{\lambda}(\overline{\theta} - \delta)}\left[\frac{\left(1 + ax^{\lambda+1}\right)^{\lambda+1}}{a(\lambda+1)^2}\right]_0^{\theta^*} \\
&= 1 - \frac{\theta^*}{\overline{\theta} - \delta} + \frac{\left(1 + a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(\overline{\theta} - \delta)(1-\delta)^{\lambda}(\lambda+1)^2}\,,
\end{aligned}$$

which proves the result in the general case. Using theorem 5, we obtain the total probability of default as:

$$\begin{aligned}
\mathbb{P}(\gamma > 1) &= \pi \cdot \mathbb{P}(\theta > \delta) \cdot \mathbb{P}(\gamma > 1\,|\,\theta > \delta) \\
&= \pi \cdot \frac{\overline{\theta} - \delta}{\overline{\theta}} \cdot \left[1 - \frac{\theta^*}{\overline{\theta} - \delta} + \frac{\left(1 + a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(\overline{\theta} - \delta)(1-\delta)^{\lambda}(\lambda+1)^2}\right]. \\
&= \frac{\pi}{\overline{\theta}} \cdot \left[\overline{\theta} - \delta - \theta^* + \frac{\left(1 + a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(1-\delta)^{\lambda}(\lambda+1)^2}\right]
\end{aligned}$$

A.2.2 The fund's expected value

We focus now on the second term of Equation (13), that is the expected liquidated proportion of risky asset in case of no failure.

Theorem 6 *The expected liquidated proportion of risky asset is available in closed-form and given by:*

$$\mathbb{E}\left[\gamma \mathbb{1}_{\{\gamma<1\}}|\theta > \delta\right] = \frac{\lambda}{(1-\delta)(1-\lambda)(\overline{\theta} - \delta)}\left[\frac{\left(1 + a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(\lambda+1)^2(1-\delta)^{\lambda-1}} - \frac{{\theta^*}^2}{2} - a\frac{{\theta^*}^{\lambda+3}}{\lambda+3}\right]. \tag{18}$$

To express the expected value of expression (13), we now need to compute the expected fund's value in the case when there is a big liquidity shock. Let us rewrite Equation for convenience (13):

$$\mathbb{P}(\gamma \leqslant 1 \,|\, \theta > \delta) \cdot \mathbb{E}\left[1-\gamma \,|\, \theta > \delta, \gamma \leqslant 1\right] = \mathbb{P}(\gamma \leqslant 1 | \theta > \delta) - \mathbb{E}\left(\gamma \mathbb{1}_{\{\gamma \leqslant 1\}} | \theta > \delta\right) .$$

The survival probability can be easily deduced from the default probability computed in the previous section. We therefore focus on the expectation term.

$$\begin{aligned}
& \mathbb{E}\left[\gamma \mathbb{1}\{\gamma < 1\} | \widetilde{\theta} > 0\right] \\
= \; & \mathbb{E}\left[\frac{\widetilde{\theta}\left(1+a\widetilde{\theta}^{\lambda+1}\right)}{1-\delta}\frac{1}{\alpha}\mathbb{1}\left\{\frac{1}{\alpha} < \frac{1-\delta}{\widetilde{\theta}\left(1+a\widetilde{\theta}^{\lambda+1}\right)}\right\} | \widetilde{\theta} > 0\right] \\
= \; & \mathbb{E}\left[\frac{\widetilde{\theta}\left(1+a\widetilde{\theta}^{\lambda+1}\right)}{1-\delta}\mathbb{E}\left(\frac{1}{\alpha}\mathbb{1}\left\{\frac{1}{\alpha} < \frac{1-\delta}{\widetilde{\theta}\left(1+a\widetilde{\theta}^{\lambda+1}\right)}\right\} | \widetilde{\theta}\right) | \widetilde{\theta} > 0\right] \\
= \; & \mathbb{E}\left[\frac{\widetilde{\theta}\left(1+a\widetilde{\theta}^{\lambda+1}\right)}{1-\delta}\int_0^{\max\left(0;\log\left(\frac{1-\delta}{\widetilde{\theta}(1+a\widetilde{\theta}^{\lambda+1})}\right)\right)} \lambda e^{(1-\lambda)x}\,\mathrm{d}x \,|\widetilde{\theta} > 0\right] \\
= \; & \mathbb{E}\left[\frac{\lambda\widetilde{\theta}\left(1+a\widetilde{\theta}^{\lambda+1}\right)}{(1-\delta)(1-\lambda)}\mathbb{1}\left\{\frac{1-\delta}{\widetilde{\theta}(1+a\widetilde{\theta}^{\lambda+1})} > 1\right\}\left[\left(\frac{\widetilde{\theta}\left(1+a\widetilde{\theta}^{\lambda+1}\right)}{1-\delta}\right)^{\lambda-1} - 1\right] |\widetilde{\theta} > 0\right] \\
= \; & \frac{\lambda}{(1-\delta)(1-\lambda)}\int_0^{\overline{\theta}-\delta}\frac{x\left(1+ax^{\lambda+1}\right)}{\overline{\theta}-\delta}\mathbb{1}\left\{x\left(1+ax^{\lambda+1}\right) < 1-\delta\right\}\left[\left(\frac{x\left(1+ax^{\lambda+1}\right)}{(1-\delta)e^{\rho_1}}\right)^{\lambda-1} - 1\right]\mathrm{d}x \\
= \; & \frac{\lambda}{(1-\delta)(1-\lambda)(\overline{\theta}-\delta)}\int_0^{\theta^*} x\left(1+ax^{\lambda+1}\right)\left[\left(\frac{x\left(1+ax^{\lambda+1}\right)}{1-\delta}\right)^{\lambda-1} - 1\right]\mathrm{d}x \\
= \; & \frac{\lambda}{(1-\delta)(1-\lambda)(\overline{\theta}-\delta)}\left\{\frac{1}{(1-\delta)^{\lambda-1}}\left[\frac{(1+ax^{\lambda+1})^{\lambda+1}}{a(\lambda+1)^2}\right]_0^{\theta^*} - \left[\frac{x^2}{2} + a\frac{x^{\lambda+3}}{\lambda+3}\right]_0^{\theta^*}\right\} \\
= \; & \frac{\lambda}{(1-\delta)(1-\lambda)(\overline{\theta}-\delta)}\left[\frac{\left(1+a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(\lambda+1)^2(1-\delta)^{\lambda-1}} - \frac{{\theta^*}^2}{2} - a\frac{{\theta^*}^{\lambda+3}}{\lambda+3}\right] .
\end{aligned} \tag{19}$$

Combining this expression with Equation (13), we obtain:

$$\begin{aligned}
& \pi \cdot \frac{\overline{\theta}-\delta}{\overline{\theta}} \cdot (1-\delta) \cdot R \cdot \mathbb{P}(\gamma \leqslant 1 \,|\, \theta > \delta) \cdot \mathbb{E}\left[(1-\gamma) \,|\, \theta > \delta, \gamma \leqslant 1\right] \\
= \; & \pi \cdot \frac{\overline{\theta}-\delta}{\overline{\theta}} \cdot (1-\delta) \cdot R \cdot \left[\frac{\theta^*}{\overline{\theta}-\delta} - \frac{\left(1+a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(\overline{\theta}-\delta)(1-\delta)^{\lambda}(\lambda+1)^2}\right. \\
& \left. - \frac{\lambda}{(1-\delta)(1-\lambda)(\overline{\theta}-\delta)}\left(\frac{\left(1+a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(\lambda+1)^2(1-\delta)^{\lambda-1}} - \frac{{\theta^*}^2}{2} - a\frac{{\theta^*}^{\lambda+3}}{\lambda+3}\right)\right] \\
= \; & \frac{\pi}{\overline{\theta}} \cdot (1-\delta) \cdot R \cdot \left[\theta^* + \frac{\lambda}{(1-\delta)(1-\lambda)}\left(\frac{{\theta^*}^2}{2} + a\frac{{\theta^*}^{\lambda+3}}{\lambda+3}\right) - \frac{\left(1+a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(1-\delta)^{\lambda}(\lambda+1)^2} \cdot \frac{1}{1-\lambda}\right]
\end{aligned} \tag{20}$$

We are now able to derive the expected fund's value in a closed-form. Using Theorems 5 and 6, we obtain the following expression for Equation (13).

$$\mathbb{E}(F) = (1-\pi)\,F_0 + \pi \cdot \frac{\delta}{\overline{\theta}} \cdot \left(F_0 - \frac{\delta}{2}\right)$$
$$+ \quad \frac{\pi}{\overline{\theta}} \cdot (1-\delta) \cdot R \cdot \left[\theta^* + \frac{\lambda}{(1-\delta)(1-\lambda)}\left(\frac{{\theta^*}^2}{2} + a\frac{{\theta^*}^{\lambda+3}}{\lambda+3}\right) - \frac{\left(1+a{\theta^*}^{\lambda+1}\right)^{\lambda+1} - 1}{a(1-\delta)^{\lambda}(\lambda+1)^2} \cdot \frac{1}{1-\lambda}\right] \tag{21}$$

where $\theta^* = \max\left\{x \in [0, \overline{\theta}-\delta] \quad \text{such that} \quad x(1+ax^{\lambda+1}) \leqslant 1-\delta\right\}$. Even though the function is ill-defined for $\lambda = 1$, it is possible to calculate it in closed-form. It can be easily shown the the fund's value for $\lambda \neq 1$ converges when λ approaches 1. A continuity argument can be applied without loss of generality. This proves Theorem 4.

To obtain the result of Equation (4), we set $a = 0$. We have:

$$\begin{aligned}
&\mathbb{E}(F) = (1-\pi)\,F_0 + \pi \cdot \frac{\delta}{\overline{\theta}} \cdot \left(F_0 - \frac{\delta}{2}\right) \\
+ \quad & \frac{\pi}{\overline{\theta}} \cdot (1-\delta) \cdot R \cdot \left[\overline{\theta} - \delta + \frac{\lambda}{(1-\delta)(1-\lambda)}\left(\frac{(\overline{\theta}-\delta)^2}{2}\right) - \frac{(\overline{\theta}-\delta)^{\lambda+1}}{(\lambda+1)(1-\delta)^{\lambda}} \cdot \frac{1}{1-\lambda}\right] \\
= \quad & F_0 + \pi \cdot \left(F_0\left(\frac{\delta}{\overline{\theta}} - 1\right) - \frac{\delta}{\overline{\theta}} \cdot \frac{\delta}{2}\right) \\
+ \quad & \frac{\pi}{\overline{\theta}} \cdot \frac{\overline{\theta}-\delta}{1-\lambda} \cdot (1-\delta) \cdot R \cdot \left[1 - \lambda + \frac{\lambda}{1-\delta}\left(\frac{\overline{\theta}-\delta}{2}\right) - \frac{(\overline{\theta}-\delta)^{\lambda}}{(\lambda+1)(1-\delta)^{\lambda}}\right] \\
= \quad & F_0 - \pi \cdot \frac{\delta^2}{2\overline{\theta}} + \frac{\pi}{\overline{\theta}}\left[F_0(\delta - \overline{\theta}) + (\overline{\theta}-\delta)(1-\delta)R\right] \\
- \quad & \frac{\pi}{\overline{\theta}} \cdot \frac{\overline{\theta}-\delta}{1-\lambda} \cdot (1-\delta) \cdot R \cdot \left[\frac{1}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{1-\delta}\right)^{\lambda} - \frac{\lambda}{1-\delta}\left(\frac{\overline{\theta}-\delta}{2}\right)\right] \\
= \quad & F_0 - \pi \cdot \delta\left(1 - \frac{\delta}{2\overline{\theta}}\right) \\
- \quad & \frac{\pi}{\overline{\theta}} \cdot \frac{\overline{\theta}-\delta}{1-\lambda} \cdot (1-\delta) \cdot R \cdot \left[\frac{1}{\lambda+1}\left(\frac{\overline{\theta}-\delta}{1-\delta}\right)^{\lambda} - \frac{\lambda}{1-\delta}\left(\frac{\overline{\theta}-\delta}{2}\right)\right].
\end{aligned}$$

Further setting $\overline{\theta} = 1$, we obtain:

$$\begin{aligned}
\mathbb{E}(F) \quad &= \quad F_0 - \pi \cdot \delta\left(1 - \frac{\delta}{2}\right) - \pi \cdot \frac{(1-\delta)^2}{1-\lambda} \cdot \left[\frac{1}{\lambda+1} - \frac{\lambda}{2}\right] \cdot R \\
&= \quad F_0 - \pi \cdot \delta\left(1 - \frac{\delta}{2}\right) - \frac{\pi}{2} \cdot (1-\delta)^2 \cdot \left[\frac{1}{\lambda+1} + 1\right] \cdot R
\end{aligned}$$

Indeed, one can check that $\frac{2}{(1-\lambda)(\lambda+1)} - \frac{\lambda}{(1-\lambda)} = \frac{1}{\lambda+1} + 1$.

A.2.3 Closed-form cash solution in the benchmark case

We hereby assume that $\overline{\theta} = 1$ and $a = 0$. Based on the computations of Appendix A.2.2, we have:

$$\begin{aligned}
\mathbb{E}(F) &= F_0 - \pi \cdot \delta \left(1 - \frac{\delta}{2}\right) - \pi \cdot \frac{(1-\delta)^2}{1-\lambda} \cdot \left[\frac{1}{\lambda+1} - \frac{\lambda}{2}\right] \cdot R \\
&= 1 + (1-\delta)(R-1) + \frac{\pi}{2} \cdot \left[(1-\delta)^2 - 1\right] - \pi \cdot \frac{(1-\delta)^2}{1-\lambda} \cdot \left[\frac{1}{\lambda+1} - \frac{\lambda}{2}\right] \cdot R \\
&= 1 - \frac{\pi}{2} + (R-1)(1-\delta) + \pi \cdot \left[\frac{1}{2} - \frac{1}{1-\lambda} \cdot \left(\frac{1}{\lambda+1} - \frac{\lambda}{2}\right) \cdot R\right] (1-\delta)^2
\end{aligned}$$

Equivalently, when $\lambda = 1$, the expected fund's value can be written as a quadratic function of δ by continuity. The program of the fund is to maximize its expected value under the constraint that $\delta \in [0, 1]$. We compute the derivative of this expectation. We trivially have:

$$\begin{aligned}
1 - \delta^* &= \frac{1}{\pi} \cdot \frac{1-R}{1 - \frac{2}{1-\lambda} \cdot \left(\frac{1}{\lambda+1} - \frac{\lambda}{2}\right) \cdot R} \\
&= \frac{1}{\pi} \cdot \frac{1/R - 1}{1/R - \frac{2}{1-\lambda} \cdot \left(\frac{1}{\lambda+1} - \frac{\lambda}{2}\right)} \\
&= \frac{1}{\pi} \cdot \frac{(\lambda+1)\left(1 - \frac{1}{R}\right)}{(\lambda+1)\frac{2}{1-\lambda} \cdot \left(\frac{1}{\lambda+1} - \frac{\lambda}{2}\right) - (\lambda+1)\frac{1}{R}} \\
&= \frac{1}{\pi} \cdot \frac{(\lambda+1)\left(1 - \frac{1}{R}\right)}{\frac{2}{1-\lambda} - \frac{\lambda(\lambda+1)}{1-\lambda} + (\lambda+1)\left(1 - \frac{1}{R}\right) - (\lambda+1)} \\
&= \frac{1}{\pi} \cdot \frac{(\lambda+1)\left(1 - \frac{1}{R}\right)}{(\lambda+1)\left(1 - \frac{1}{R}\right) + \frac{2-\lambda(\lambda+1)}{1-\lambda} - \frac{1-\lambda^2}{1-\lambda}} \\
&= \frac{1}{\pi} \cdot \frac{(\lambda+1)\left(1 - \frac{1}{R}\right)}{(\lambda+1)\left(1 - \frac{1}{R}\right) + \frac{1-\lambda^2-\lambda+\lambda^2}{1-\lambda}} \\
&= \frac{1}{\pi} \cdot \frac{(\lambda+1)\left(1 - \frac{1}{R}\right)}{(\lambda+1)\left(1 - \frac{1}{R}\right) + 1} \blacksquare
\end{aligned}$$

This proves Theorem 1.

The expected value is given by:

$$\mathrm{EF}^* = 1 - \frac{\pi}{2} - \frac{(R-1)^2}{4\pi \cdot \left[\frac{1}{2} - \frac{1}{1-\lambda} \cdot \left(\frac{1}{\lambda+1} - \frac{\lambda}{2}\right) \cdot R\right]} \tag{22}$$

Thus, using the fact that $\frac{2}{(1-\lambda)(\lambda+1)} - \frac{\lambda}{(1-\lambda)} = \frac{1}{\lambda+1} + 1$

$$\begin{aligned} \text{EF}^* &= 1 - \frac{\pi}{2} - \frac{(R-1)^2}{2\pi \cdot \left[1 - \left(\frac{1}{\lambda+1} + 1\right) \cdot R\right]} \\ &= 1 - \frac{\pi}{2} - \frac{(R-1)^2}{2\pi \cdot \left[1 - R - \frac{R}{\lambda+1}\right]} \end{aligned}$$

Given the optimal quantity of cash δ^*, the total default probability is given by (see Equation (17)):

$$\begin{aligned} \mathbb{P}(\gamma > 1) &= \pi \cdot \frac{1-\delta^*}{\lambda+1} \\ &= \pi \cdot \frac{1-\frac{1}{R}}{(\lambda+1)\left(1-\frac{1}{R}\right)} \\ &= \frac{1}{\pi} \cdot \frac{1}{\lambda+1} \end{aligned}$$

This proves Theorem 2.

A.2.4 Risk premium analysis

We hereby assume that $\overline{\theta} = 1$, $a = 0$ and $R = 1 + \frac{\kappa}{\lambda+1-\kappa}$. We can easily derive the different quantities. The fund's optimal cash decision is given by:

$$\begin{aligned} 1-\delta^* &= \frac{1}{\pi} \cdot \frac{(\lambda+1)\left(1-\frac{1}{R}\right)}{(\lambda+1)\left(1-\frac{1}{R}\right)+1} \\ &= \frac{1}{\pi} \cdot \frac{(\lambda+1)\frac{\kappa}{\lambda+1}}{(\lambda+1)\frac{\kappa}{\lambda+1}+1} \\ &= \frac{1}{\pi} \cdot \frac{\frac{\kappa}{\lambda+1}}{\frac{\kappa+1}{\lambda+1}} \\ &= \frac{1}{\pi} \cdot \frac{\kappa}{\kappa+1} \end{aligned}$$

The expected fund value at the optimum is:

$$\begin{aligned} \text{EF}^* &= 1 - \frac{\pi}{2} + \frac{\left(\frac{\kappa}{\lambda+1-\kappa}\right)^2}{2\pi \cdot \left[\frac{\kappa}{\lambda+1-\kappa} + \frac{1}{(\lambda+1-\kappa)(\lambda+1)}\right]} \\ &= 1 - \frac{\pi}{2} + \frac{\kappa^2}{2\pi \cdot (\lambda+1-\kappa)} \cdot \frac{1}{\kappa + \frac{\kappa+\lambda+1-\kappa}{\lambda+1}} \\ &= 1 - \frac{\pi}{2} + \frac{\kappa^2}{2\pi \cdot (\lambda+1-\kappa)(\kappa+1)} \end{aligned}$$

and the liquidation probability is independent from R so it stays the same. This proves Theorem 3. Putting $\kappa = 1$, we have:

$$\begin{aligned} 1 - \delta^* &= \frac{1}{2\pi} \\ \mathrm{EF}^* &= 1 - \frac{\pi}{2} + \frac{1}{4 \cdot \pi \cdot \lambda} \end{aligned}$$

This proves Corollaries 1.1 and 2.1.